ON THE CONDITIONS FOR THE EXISTENCE OF FRACTAL DOMAIN INTEGRALS

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It is shown that noise transformation processes in some systems with delay may be described in terms of two sorts of fractal domain integrals. The convergence conditions of these integrals are analyzed. The structure of multifractals associated with the integrals for different contraction coefficients is considered.

1. Introduction

Extensive investigations of self-similar structures (the "physical" fractals) have led to numerous publications in recent years (see [1, 2] for a recent review). These investigations were stimulated to a certain extent by a number of previous theoretical works which explained and popularized the notions of fractal geometry (note the pioneer works of Mandelbrot [3, 4], and also [5-8]). One of the key ideas was to relate a natural object to a nonstandard set of fractional dimension (a "mathematical" fractal), similar to the Cantor set, or to a singular distribution of some quantity over such a set (i.e., the multifractal measure or the "multifractal" [9]). The Cantor fractal thus acquires the status of an idealized model of a real medium, with averages over the fractal playing the role of physical quantities. Although this theoretical scheme cannot be objected to, in general, it should nevertheless be applied with some caution. One should take into account that an infinite extension of the hierarchy of self-similarity levels may cause convergence problems for the average values. In that case, the computations performed to the "physical" standards of rigorousness would be fraught with erroneous results. We shall demonstrate this in the present paper by considering a simple integral over the multifractal (MFI) and deriving the conditions for its convergence.

Define the MFI of \( f(x) \), with \( x \in [0, 1] \), as

\[
\int_{\mathcal{L}} f(x) d\mu(x|\kappa, \Theta) = \lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} \sigma_n^#,
\]

where

\[
\sigma_n = 2^{-n} \sum_s \Theta_n^{[s]} f(\lambda_n^{[s]}),
\]

\[
\sigma_n^# = 2^{-n} \sum_s \Theta_n^{[s]} \langle f(\lambda_n^{[s]}) \rangle_{\kappa_n},
\]

(1)

(2)

(3)

(the conditions for the limits in (1) to exist, and to be equal, will be derived in Sec. 3). In this formula. \( s_n = (s_n^{(1)}, \ldots, s_n^{(n)}) \) is a signature vector with components \( s_n^{(i)} = \pm 1 \); summation in (2) and (3) goes over the set of such vectors. The argument of the function takes on the following values:

\[
\lambda_n^{[s]} = \frac{1}{2} + \frac{1}{2} (1 - \kappa)(s_n, h_n(\kappa)),
\]

(4)

where \( h_n = (1, \kappa, \kappa^2, \ldots, \kappa^{n-1}) \), \( \kappa < 1 \). The symbol \( (\mathbf{a}, \mathbf{b}) \) is used for the scalar product. The average is taken in (3) according to the formula

\[
\langle f(x) \rangle_\delta = \frac{1}{\delta} \int_{x-\delta/2}^{x+\delta/2} f(t) dt.
\]

(5)
The quantities \( \Theta_{n}^{[s]} \) are functions of \( s \). In what follows, we shall assume the normalization condition 
\( \sigma_{n}^{\#} = 1 \) for \( f \equiv 1 \). Then it follows that
\[
\Theta_{n}^{[s]} + \Theta_{n}^{[s']} = 2\Theta_{n-1}^{[s]}
\]
for \( s_n = s_{n-1} \oplus (+1) \), \( s_{n} = s_{n-1} \oplus (-1) \) (with the \( \oplus \) operation being the “concatenation” of vectors).

We shall say that the MFI (1) is of the first class if the conditions
\[
\begin{align*}
\text{(7a)} & & \Theta_{n}^{[s]} = \prod_{i=1}^{n} \psi(s_{n}^{(i)}), & & \psi(+1) + \psi(-1) = 2, \\
\text{(7b)} & & \text{Im} \psi(\pm 1) = 0, & & \text{Re} \psi(\pm 1) > 0
\end{align*}
\]
are fulfilled: a MFI is said to be of the second class if Eqs. (7a) hold while at least one of the conditions (7b) is violated. In the first case, one can interpret \( p_{\pm} = \frac{1}{2} \psi(\pm 1) \) as probabilities; the corresponding multifractal measures induced by the multiplicative Besicovitch process were considered in [9].

It will be shown in Sec. 2 that both first- and second-class MFI appear in the problem of stationary noise in systems with delay. In Sec. 3, we find the conditions for the integrals to converge, while in Sec. 4, we consider the structure of the fractal domain for various \( \kappa \).

2. Noise transformation in systems with delay

2.1. Linear system with dichotomous and Gaussian noise.

Let us assume that the transformation of a two-dimensional (complex) noise in a linear system with delay is described by the difference equation
\[
X(t) = \kappa X(t - \tau) + \xi(t - \tau)
\]
for \( \kappa \in \mathbb{R} \), \( \kappa < 1 \), with \( \xi = \xi_{g} + \xi_{d} \) where \( \xi_{g} \) is the Gaussian stochastic process (SP) and \( \xi_{d} \) is the dichotomous SP (a particular case of the Kubo–Andersen SP [10, 11]). If the noise correlation time satisfies the condition \( \tau_{\text{cor}} \ll \tau \), the partition function of \( X \) satisfies the Kolmogorov–Chapman type equation [12],
\[
P_{N+1}(X) = \int dY \{ P_{g} * P_{d} \}(X - \kappa Y) P_{N}(Y),
\]
where
\[
P_{g}(X, R) = (\pi R)^{-1} e^{-|x|^{2}/R}, \quad P_{d}(X) = \sum_{i=1}^{2} p_{i} \delta^{(2)}(X - A_{i});
\]
\[
P_{N}(X) = P(t_{0} + N\tau, X), \quad t_{0} \in [0, \tau), \quad N \in \mathbb{N};
\]
\[
\delta^{(2)}(X) = \delta(\text{Re} X)\delta(\text{Im} X); \quad R = \langle |\xi_{g}|^{2} \rangle,
\]
and \( * \) denotes convolution. Going to the equation for characteristic functions and solving it by iteration, it is not difficult to represent the stationary solution in the form
\[
P_{\text{st}}(X) = \lim_{N \to \infty} P_{N}(X) = \int_{C} P_{g} \left( X - \frac{A_{1}x + A_{2}(1 - x)}{1 - \kappa}, \frac{R}{1 - \kappa^2} \right) d\mu(x|\kappa, \Theta).
\]
The expression thus obtained involves a first class MFI: \( \frac{1}{2} \psi(+1) = p_{1}, \frac{1}{2} \psi(-1) = p_{2} \).