ANALOGS OF FOURIER SERIES FOR A RELATIVISTIC STRING MODEL WITH MASSIVE ENDS

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A description of the motions of a relativistic string with massive ends (for the model with a finite mass on the first end and an infinite mass on the second end) is proposed. This approach uses the expansion of the string world surface into a series that is not reduced to the ordinary Fourier series due to the nonlinearity of the problem. The state equation of the string is derived from the mass shell condition for its end. The string motions are classified, allowing linearization of the boundary condition by a natural parametrization of the trajectory of the moving end. The set of such world surfaces is shown to be limited; for the special cases of 2 + 1- and 3 + 1-dimensional Minkowski spaces, all of them reduce to a helicoid.

Introduction

The model of a relativistic string with masses on its ends [1–6] used to describe a system of two quarks coupled by strong interaction was one of the reasons for the intense development of the string theory that is rather branched these days (see review [7]). However, progress in the theory of strings with massive ends is held back by the essential difficulties which arise when we pass to a quantum description, due to the nonlinearity of the model.

The main point of the problem is that the string theory is developed as a two-dimensional field theory, in which this field, with some coupling, satisfies the string oscillation equation—a two-dimensional analog of the wave equation [1–8]. In the models of closed and free (with massless ends) strings, the field defined by the coordinate of the string point $X^\mu$ obeys the linear boundary conditions that allow the field to expand with respect to two-dimensional plane waves. Mathematically, this is equivalent to the expansion of the periodic functions of one argument into a Fourier series. When quantified, the amplitudes of the harmonics in this series are identified with quantum operators. In models of a string with massive ends, the boundary conditions are essentially nonlinear in form [1–4]. Consequently, the sum of two functions satisfying these equations does not, in general, satisfy these equations. This contradiction with the quantum superposition principle does not allow the above-mentioned quantization scheme, using the expansion of $X^\mu$, to be applied here.

In the present paper, we introduce some analogs of Fourier series that make it possible to represent the world surface $X^\mu$ as defined by nonlinear boundary conditions. In the first section, the main equations that describe our model are discussed, as well as the relations obtained in [5], [9] concerning the unity vector of the string end velocity. This value, which specifies the string world surface with an accuracy up to the translations, plays a special role in our approach. The latter is described in Sec. 2 as an example of a string with an infinitely heavy (fixed) end. In particular, our model describes the centrosymmetric motion of a string with identical masses at its ends. In Sec. 3, a class of string motions is presented that allows the natural parametrization to be described for the trajectory of its movable end.

1. Equations of motion and boundary conditions for a relativistic string

The motion of a relativistic string with tension $\gamma$ and masses $m_1$, $m_2$ on its ends is described by the
Here $X^\mu(\tau, \sigma)$ are the coordinates of the string point in $D$-dimensional Minkowski space $\mathbb{R}^{1,D-1}$, $\sigma_1(\tau)$ and $\sigma_2(\tau)$ are the inner coordinates of the string ends, $X^\mu = \partial X^\mu/\partial \tau$, $X'^\mu = \partial X'^\mu/\partial \sigma$; $(ab) = a^\mu b^\nu = \eta_{\mu\nu}a^\mu b^\nu$ are the pseudo(scalar) products in $\mathbb{R}^{1,D-1}$, and the velocity of light is $c = 1$.

Variation of action (1.1) yields equations for the string motion and boundary conditions, which have their simplest form if by the choice of coordinates $\tau$ and $\sigma$ we make the induced metric on the string world surface conformally flat, i.e., satisfying the conditions of the orthonormal gauge

\begin{equation}
\sum_{a=1}^{2} m_a \sqrt{\left( \frac{d}{d\tau} X^\mu(\tau, \sigma_a(\tau)) \right)^2} = 0.
\end{equation}

When these conditions are satisfied, the equations of motion become linear,

\begin{equation}
\ddot{X}^\mu - X'^{\mu} = 0,
\end{equation}

and the boundary conditions at the string ends numerated by index $a$ are of the form

\begin{equation}
\left. \left\{ m_a \frac{d}{d\tau} \frac{X^\mu + \sigma'_a(\tau) X'^\mu}{\sqrt{X^2 - (1 - \sigma'_a(\tau)^2)}} + (-1)^a \gamma \left[ X'^\mu + \sigma'_a(\tau) \dot{X}^\mu \right] \right\} \right|_{\sigma = \sigma_a(\tau)} = 0.
\end{equation}

Let us transform the boundary conditions (1.4) by substituting the general solution (1.3):

\begin{equation}
X^\mu(\tau, \sigma) = \frac{1}{2} \left[ \psi^\mu_+(\tau + \sigma) + \psi^\mu_-(\tau - \sigma) \right].
\end{equation}

Derivatives of the functions introduced here are isotropic because of the conditions (1.2)

\begin{equation}
\psi^{-2}_-(\xi) = \psi^{+2}_+(\xi) = 0.
\end{equation}

We fix the inner equations of the string-end trajectories in the form

\begin{equation}
\sigma_1(\tau) = 0, \quad \sigma_2(\tau) = \pi,
\end{equation}

which can always be done [1.4] in view of the invariance of expressions (1.2)

\begin{equation}
\tilde{\tau} \pm \tilde{\sigma} = f_{\pm}(\tau \pm \sigma).
\end{equation}

Note that conditions (1.7) do not completely fix the freedom of choice for functions $f_{\pm}(\xi)$. If $\phi(\xi)$ is an arbitrary $2\pi$-periodic smooth function with bounded derivatives, then changes of variables of the class (1.8) with

\begin{equation}
f_+(\xi) = f_-(\xi) = \xi + \phi(\xi), \quad \phi(\xi + 2\pi) = \phi(\xi), \quad |\phi'(\xi)| < 1,
\end{equation}

preserve the form of (1.7) of the string end equations $\tilde{\sigma} = 0$ and $\tilde{\sigma} = \pi$. The arbitrariness of the choice of coordinates (1.9) on the world surface (1.5) can be interpreted as the physical insignificance of the "longitudinal oscillations" of the inner string points (these points are indistinguishable with respect to action (1.1)).