EXACT SOLUTION OF THE BETHE–SALPETER EQUATION WITH RETARDING PROPAGATORS IN THE WICK–CUTKOSKY MODEL

V. I. Savrin and A. Yu. Troole

We study the discrete spectrum of the Bethe-Salpeter equation in the Wick-Cutkosky model and show that for retarding propagators, the variables can be separated, and in the S-wave case, the equation can be reduced to a one-dimensional equation. By using the method of contour integration, we obtain the exact solution of this equation and study the spectrum of relativistic bound states. We discuss the ambiguity of the analytic expression of the wave function that is caused by the presence of a nonphysical variable in the four-dimensional description of relativistic systems.

Introduction

The method of contour integration was proposed in [1] for obtaining exact solutions of the integral equation describing a bound system of two relativistic particles in momentum space. As such an equation, the quasi-potential Logunov-Tavkhelidze equation [2] has been considered. This equation does not contain any nonphysical parameters of the relative energy of particles and, thus, is three-dimensional. As the quasi-potential, the contribution of the one-gluon exchange to the interaction of two scalar particles was taken.

Since the proposed method is efficient, it can be used to solve the Bethe–Salpeter equation for the Wick-Cutkosky model, which, it is possible to say, simulates the scalar Coulomb interaction in quantum field theory.

The Wick–Cutkosky model [3], which describes the interaction of two scalar particles via the massless scalar boson in the ladder approximation, is well known. The last review of results obtained within this model was published by Nakanishi [4]. This is a unique model. Being a relativistic two-body quantum problem, it can be solved analytically and, as a matter of fact, it is ideal for studying various properties of a two-particle system.

Unfortunately, no analytic expression for the energy levels of the system has yet been obtained within the limits of this model. Although there are different methods for reducing the four-dimensional integral equation to a one-dimensional second-order differential equation [4], the indeterminacy in the boundary conditions results in different boundary value problems whose analytic solutions are still unknown. The indeterminacy follows from the general problem of stating physically acceptable boundary conditions for different, explicitly covariant differential equations. These equations pretend to describe the bound states of different quantum field systems, since the relative energy or the relative time of two particles is one of the four explicit variables in these equations (it is difficult to pose boundary conditions for these variables without any physical assumptions).

Since the integral equation implicitly contains the necessary boundary conditions, it is natural to try to solve this equation directly. Unfortunately, even in the S-wave case, no analytic solution of the initial equation has yet been obtained. However, if we replace all causal propagators in the initial equation by retarding ones, then, as shown below, we can solve the modified equation exactly and completely investigate the spectral properties of the corresponding integral operator in the complex plane of values of the energy of the system.

Naturally, there is an immediate question about the relation between the equation with retarding propagators and the equation with causal propagators. We discuss this question in Sec. 4.
1. Separation of variables

In the Wick–Cutkosky model, we write the Bethe–Salpeter equation for bound states in an immovable frame of reference and restrict our consideration to the case of two particles of equal mass \( m \):

\[
\frac{1}{4\pi^2} \int d^4k \Phi(k) \frac{\Phi(p) \delta\left(p^{-}\!-\!\frac{1}{2}(p^{+}-M)\right)}{(p-k)^2 + i0} = \frac{i\lambda}{\pi^2} \int \frac{d^4k}{(p-k)^2 + i0},
\]

where \( 2M = \sqrt{\frac{2}{3}} \) is the mass of the bound system, \( \lambda = g^2/(4\pi)^2 \), and \( g \) is the coupling constant in our theory.

By passing to retarding propagators, we change the method for going around the poles. As is easy to see, the equation with retarding propagators has the form

\[
\frac{1}{4\pi^2} \int d^4k \Phi(k) \frac{\Phi(p) \delta\left(p^{-}\!-\!\frac{1}{2}(p^{+}-M)\right)}{(p-k)^2 + i0} = \frac{i\lambda}{\pi^2} \int \frac{d^4k}{(p-k)^2 + i0},
\]

We shall consider any complex variable \( M \) with a positive imaginary part. We also note that Eq. (2) holds only for a particle–antiparticle system.

In the present paper, we consider only equations for the \( S \)-wave. This means that in (2), \( \Phi(p) = \Phi(p_0, p^\pm) \) is independent of the angles. In what follows, we shall write \( p \) instead of \( |p| \) and hope that this does not lead to a misunderstanding. We integrate over the angles and introduce the function \( \Phi(p_0, p) = \Phi(p_0, \{p^\pm\}) \).

Then we have

\[
\frac{1}{4\pi^2} \int d^4k \Phi(k) \frac{\Phi(p) \delta\left(p^{-}\!-\!\frac{1}{2}(p^{+}-M)\right)}{(p-k)^2 + i0} = \frac{i\lambda}{\pi^2} \int \frac{d^4k}{(p-k)^2 + i0},
\]

Let us continue \( \Phi(p_0, p) \) to the negative half-axis antisymmetrically in \( p \): \( \Phi(p_0, p) = -\Phi(p_0, -p) \). Passing to the wave-front variables, \( p_0 + p = p^+, \ p_0 - p = p^- \), \( \Phi(p^+, p^-) = -\Phi(p^-, p^+) \), we obtain the equation

\[
\frac{1}{4\pi^2} \int d^4k \Phi(k) \frac{\Phi(p) \delta\left(p^{-}\!-\!\frac{1}{2}(p^{+}-M)\right)}{(p-k)^2 + i0} = \frac{i\lambda}{\pi^2} \int \frac{d^4k}{(p-k)^2 + i0},
\]

which can be separated into the wave front variables.

We introduce the notation

\[
F(z) = \int d^4z \int d^4z' \Phi(z') \log \frac{z - z' + i0}{z - z'' + i0},
\]

and for \( \Phi(p^+, p^-) \), we have the representation

\[
\Phi(p^+, p^-) = \frac{\lambda}{4i\pi} \int d^4k \Psi(k^+, k^-) \left\{ \log \frac{p^+ - k^+ + i0}{p^+ - k^- + i0} - \log \frac{p^- - k^+ + i0}{p^- - k^- + i0} \right\},
\]

Substituting (6) into (4), we obtain the equation for \( F(z) \):

\[
F(z) = \frac{\lambda}{2i\pi} \int d^4z \int d^4z' \Phi(z') \log \frac{z - z' + i0}{z - z'' + i0} = \frac{\lambda}{2M} \int d^4z \int d^4z' \Phi(z') \log \frac{z - z' + i0}{z + M - \frac{m^2}{z + M + i0}}.
\]