ON QUANTIZATION OF SYSTEMS WITH ACTIONS UNBOUNDED FROM BELOW

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We consider two possible approaches to the problem of the quantization of systems with actions unbounded from below: the Borel summation method applied to the perturbation expansion in the coupling constant and the method based on the kened Langevin equation for stochastic quantization. In the simplest case of an anharmonic oscillator, the first method produces Schwinger functions, even though the corresponding path integral diverges. The solutions of the kened Langevin equation are studied both analytically and numerically. The fictitious time averages are shown to have limits that can be considered as the Schwinger functions. The examples demonstrate that both methods may give the same result.

1. Introduction

Let \( G_1(x), G_2(x_1, x_2), \ldots, G_n(x_1, \ldots, x_n), \ldots \) be the Schwinger functions of some quantum field. In practice, they are defined by the path integrals as follows:

\[
G_n(x_1, \ldots, x_n) = \mathcal{N} \int \varphi(x_1) \ldots \varphi(x_n) e^{-A[\varphi]} \mathcal{D}\varphi,
\]

(1)

where \( \mathcal{N} \) is the normalizing factor and \( A[\cdot] \) is the Euclidean action functional.

For example, for the scalar field \( \varphi \),

\[
A[\varphi] = \frac{1}{2} \int \left[ (\partial_{\mu} \varphi)^2 + m^2 \varphi^2 \right] dx + \frac{\lambda}{4} \int \varphi^4 dx.
\]

(2)

Among the various mathematical problems involved in path integral calculation (2), there is a special problem that appears when \( \lambda \) is negative. There, action \( A[\cdot] \) is not bounded from below and, obviously, the path integral (1) diverges. Systems with such action functionals are called bottomless.

In classical mechanics, there are many models where the energy is unbounded from below which also possess regular dynamic behavior. The quantum Heisenberg description of the related models also has no principal drawbacks. Thus, one can infer that Schwinger functions might exist (in some sense) for systems with actions unbounded from below. In this paper, we discuss the two following approaches to bottomless systems:

1. Analytic continuation in \( \lambda \);
2. Stochastic quantization.

The perturbation expansion in \( \lambda \) for \( G_n \) is well defined in every order, irrespective of the sign of the coupling constant. As the perturbation series is apparently divergent, we apply the Borel method to sum the asymptotic expansions.

The simplest stochastic quantization scheme is not applicable in our case because the solutions of the corresponding Langevin equation blow up, i.e., go to infinity during some finite time period. Therefore, we

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use the ke kernel Langevin equation instead. In this paper, we restrict ourselves to a toy model, considering the \( N = 1 \) matrix model in \( D = 0 \) configuration space and we employ both of the above methods to calculate the Schwinger functions. Then the path integral (1) is reduced to an ordinal integral of the form

\[
G_n = N \int_{-\infty}^{\infty} \varphi^n \exp \left( -\frac{\varphi^2}{2} - \lambda \frac{\varphi^4}{4} \right) d\varphi. 
\]  

(3)

The paper is organized as follows. In Secs. 2 and 3, we discuss the methods of Borel summation and stochastic quantization. Next, we present the results of computer simulations (Sec. 4) and, finally, (Sec. 5) the concluding remarks.

2. Borel summation method

The idea of applying the Borel summation method to divergent series of perturbation theory first appeared in [1], where an anharmonic oscillator was studied. Later, it turned out that some asymptotic expansions in the \( P(\varphi)_2 \)-theory in infinite volumes are Borel-summable [2]. Borel summability for the Zeemann effect was proved in [3]. Here we revisit this subject for the bottomless case.

Let us consider a power series in \( \lambda \), formally defined by the integral

\[
S_{2k}(\lambda) = \int_{-\infty}^{\infty} \varphi^{2k} \exp \left( -\frac{\varphi^2}{2} - \lambda \frac{\varphi^4}{4} \right) d\varphi.
\]

It is easy to check that

\[
S_{2k}(\lambda) \sim \sum_{n=0}^{\infty} (-1)^n a_n(k) \lambda^n, 
\]

(4)

where

\[
a_n(k) = \sqrt{2\pi} \frac{1}{n!} \frac{(4n + 2k - 1)!!}{2^{2n}}.
\]

Following Borel, one defines an auxiliary function as

\[
h_k(u) = \sum_{n=0}^{\infty} a_n(k) u^n. 
\]

(5)

Then the sum of the asymptotic expansion is

\[
S_{2k}(\lambda) = \int_{0}^{\infty} e^{-\nu} h_k(-\lambda \nu) d\nu. 
\]

(6)

This expansion of function \( h_k(u) \) converges inside the circle \(|u| < \frac{1}{4}\). We need to know, however, the function \( h_k(u) \) for all real \( u > 0 \) when \( \lambda < 0 \), and for all \( u < 0 \) when \( \lambda > 0 \), in order to calculate \( S_{2k} \) by (6). This means that we are to continue this function analytically outside the circle of convergence.

Inside the circle \(|u| < \frac{1}{4}\), power series (4) converges to

\[
h_k(u) = \sqrt{2\pi} (2k - 1)!! F \left( \frac{2k + 1}{4}, \frac{2k + 3}{4}; 1; 4u \right),
\]

(7)

where \( F(\alpha, \beta; \gamma; z) \) is the Gaussian hypergeometric function (see Sec. 2.1.1 of [4]). This function is analytical in \( z \) in the complex plane with the cut \( \text{Re} \ z > 1, \text{Im} \ z = 0 \).

To obtain \( S_{2k}(\lambda) \), one has to calculate the integral

\[
S_{2k}(\lambda) = \sqrt{2\pi} (2k - 1)!! \int_{0}^{\infty} e^{-\nu} F \left( \frac{2k + 1}{4}, \frac{2k + 3}{4}; 1; -4\lambda \nu \right) d\nu. 
\]

(8)