ANALYSIS OF THE CHIRAL ANOMALY IN DIMENSIONAL
REGULARIZATION BY THE PROJECTION TECHNIQUE

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A self-consistent method for calculating the chiral anomaly in dimensional regularization without the four-dimensional symbols $\gamma_5$ and $\varepsilon_{\mu\nu\lambda\rho}$ is proposed. The method is applied to the calculation and analysis of chiral symmetry in massless quantum electrodynamics.

1. Introduction

It is generally accepted that dimensional regularization with minimum subtraction (MS) is one of the most convenient computational schemes. At the same time, it is known that in problems with spinor fields, dimensional regularization causes some difficulties if one has to deal with $\varepsilon_{\mu\nu\lambda\rho}$ and $\gamma_5 = \varepsilon_{\mu\nu\lambda\rho} \gamma_\mu \gamma_\nu \gamma_\lambda / 4!$. The simplest example is the calculation of the coefficient $a$ of the axial anomaly

$$\partial_\mu J_5^\mu = a \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}$$

in massless quantum electrodynamics (the objects in Eq. (1) are understood as renormalized composite operators). The literature offers many ways of calculating the axial anomaly in dimensional regularizations [1-3]. In this article, we suggest another, which, in our opinion, is a natural and simple way. The main idea consists of the fact that chiral transformations in the dimension $D = 4$ can be naturally extended to an arbitrary dimension $D$ without the symbols $\varepsilon_{\mu\nu\lambda\rho}$ and $\gamma_5$. This allows all calculations to be performed by the usual rules. Calculation by the MS scheme with a two-loop accuracy gives

$$a = N u + 4 N u^2 + O(u^3),$$

where $u = e^2 / 16\pi^2$ is the charge and $N$ is the number of isotopic components of the spinor field. The two-loop contribution (and, probably, all subsequent ones) is present in Eq. (2), in contrast to the formula in [1]. This is connected with the difference in the subtraction schemes. Answer (2) was obtained by analysis of the renormalized Ward identity (note that only composite operators, not their matrix elements, were used) and verified by renormalization-group equations in the spirit of [2]. The projection-technique formulas for composite operators play an important role in obtaining identity (1) for $D = 4$ [3, 5, 6].

2. Chiral transformation in dimension $D = 4 - 2\varepsilon$ and Ward identities

Let us consider Euclidean massless $U_N$-symmetric quantum electrodynamics in the dimension $D = 4 - 2\varepsilon$ and write the unrenormalized density of the Lagrangian $\mathcal{L}_0 = \mathcal{L}_0(\phi, \varepsilon_0, \alpha_0)$ in the form

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha_0} (\partial_\mu A_\mu)^2 + \bar{\psi}^a (\not\partial + ie_0 \not{A}) \psi^a,$$

where $\phi = (\bar{\psi}, \psi, A_\mu)$ is the set of all fields, $e_0$ and $\alpha_0$ are the bare parameters, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\not\partial = \frac{1}{2} (\not\partial + \not\partial)$. In what follows, we drop the isotopic index $a = 1, \ldots, N$ of the spinor fields and assume

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summation over that index in all squared forms of $\bar{\psi} \cdots \psi$. Since the symbols $\epsilon_{\mu\nu\lambda\rho}$ and $\gamma_5$ are absent in Eq. (3), the change of $D$ does not create any problems with the $\gamma$-matrices. Model (3), in the dimension $D = 4 - 2\varepsilon$, is multiplicatively renormalized and its renormalized action has the standard form

$$S_R(\phi) = \int d^D x \mathcal{L}_0(\phi Z_\phi, e_0 = e\mu Z_\varepsilon, \alpha_0 = \alpha Z_\alpha),$$

where $\mu$ is the renormalization mass, $e$ and $\alpha$ are the renormalized parameters, and $Z_{\phi,e,\alpha}$ are the renormalization constants with $Z_A = Z_\varepsilon^{-1}$ and $Z_\alpha = Z_\alpha^2$ by virtue of gauge invariance [3, 7].

For $D = 4$, the symbols $\epsilon_{\mu\nu\lambda\rho}$ and $\gamma_5 = \epsilon_{\mu\nu\lambda\rho} \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho / 4!$ make sense and model (3) is invariant under global chiral transformations of the spinor fields $\psi \rightarrow (\exp \omega \gamma_5) \psi$, $\bar{\psi} \rightarrow \bar{\psi} (\exp \omega \gamma_5)$. The infinitesimal form of these transformations is

$$\delta_\omega \psi = \omega \gamma_5 \psi, \quad \delta_\omega \bar{\psi} = \bar{\psi} \omega \gamma_5,$$

where $\omega$ is an infinitesimal parameter.

First, we want to generalize transformation (5) to the case of the varying dimension $D = 4 - 2\varepsilon$. Recall the known technique of working with $\gamma$-matrices in varying dimensions $D$ (it would be more correct to speak about "$\gamma$-symbols"). Then, one can simply use the main anticommutative relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu},$$

that allows any product of $\gamma$-matrices to be reduced to a linear combination of "base structures," the completely antisymmetrized products

$$\gamma^{(n)}_A = [\gamma_{\mu_1} \cdots \gamma_{\mu_n}],$$

where $[\cdots]$ denotes complete antisymmetrization in permutations of all indices inside the brackets and $A = (\mu_1 \cdots \mu_n)$ is the compact notation for the entire set of indices.

It should be emphasized that the objects in Eq. (7), unlike $\gamma_5$, can be assumed to be well defined in the varying dimension $D$ and the rules for handling them are well known [5, 8]. In the integer dimension $D = 4$, only structures with $n \leq 4$ in Eq. (7) are different from zero: all subsequent structures disappear. In what follows, having in mind the case $D = 4 - 2\varepsilon$, we call structures in Eq. (7) with $n \leq 4$ "junior" and structures with $n > 4$ "senior". We use similar terms for the composite operators of type $\bar{\psi} \cdots \psi$ containing expressions (7). By senior, we mean the operators formally disappearing at $D = 4$ due to the disappearance of the senior structures (7) entering them.

The following transformations can be considered as the generalization of Eq. (5) to the case $D = 4 - 2\varepsilon$:

$$\delta_\omega \psi = \omega A \gamma^{(4)}_A \psi, \quad \delta_\omega \bar{\psi} = \omega A \bar{\psi} \gamma^{(4)}_A,$$

with an infinitesimal parameter $\omega_A$ completely antisymmetric in the set of four indices $A = (\mu_1 \mu_2 \mu_3 \mu_4)$. It is clear that for $D = 4$, transformations (8) are reduced to Eq. (5), but they have meaning in the noninteger dimension, as well, though the corresponding symmetry in Eq. (3) is obviously broken.

Irrespective of the existence of symmetry, the Noether current corresponding to transformations (8) can be constructed by the given density of the Lagrangian $\mathcal{L}$. For density (3), we obtain

$$J^\nu_A = \frac{1}{2} \bar{\psi} \gamma^{(4)}_A \psi \equiv 4 \bar{\psi} (\delta_\nu [\gamma_{\mu_1} \gamma^{(3)}_{\mu_2 \mu_3 \mu_4}] \psi,$$

and the factor $Z_\gamma^2$ appears in the renormalized model (4). Note that the explicit form of the current depends on the arbitrariness of the type of total derivative in $\mathcal{L}$, which does not affect the action functional. Current (9) is gauge-invariant and $C$-even. To ensure the latter property, the antisymmetrization $\tilde{\partial} \rightarrow \partial$ was performed in Eq. (3).