DYNAMICAL STOCHASTICITY AND SPECTRUM

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The question of what are the characteristic features of the quantum spectrum of a dynamic stochastic system is studied by treating concrete examples.

1. The literature often supports the statement that the spectra of stochastic systems differ from the spectra of integrable ones in two basic features:

   (1) since stochastic systems do not have any integrals of motion other than energy, all energy levels of a stochastic system are nondegenerate;

   (2) the spectrum of the stochastic system is irregular if one is looking at the dependence of energy on the ordinal number of the state.

   The purpose of this paper is to see whether the above statements are true by looking at a simple example. To this end, we consider the Matinyan model [1–3], which is a mechanical system with two degrees of freedom with the Hamiltonian

   \[ H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2 y^2}{2}. \]  

   This equation was investigated by the author in [4].

2. The equipotential lines of this model are isoscale hyperbolas in all four Cartesian quadrants. This system is completely stochastic; its only integral of motion is the energy \( E \). However, it also has certain (unstable and with zero measure in the initial conditions) closed trajectories, some of which are symmetric.

   It is appropriate to speak of the “mode \((m, n)\)” if the trajectory closes after \( m \) “oscillations” in one variable and \( n \) in another. These modes were approximately integrated in [4]. Furthermore, the classical description allows infinite motion along the \( x \) and \( y \) axes; however, it is natural to expect that only the discrete spectrum will survive after quantization, since infinite motion is possible only if one of the coordinates equals zero, which is prohibited in the quantum description by the uncertainty relation at \( E < \infty \).

   Closed trajectories are quantized according to Bohr and Sommerfeld [4]. This leads to the following spectrum:

   \[ E = \left( \frac{3\pi}{16\sqrt{2}} \right)^{\frac{3}{2}} \hbar^\frac{3}{2} (2n_x \cdot 2n_y)^{\frac{3}{2}} = \left( \frac{3\pi}{16\sqrt{2}} \right)^{\frac{3}{2}} \hbar^\frac{3}{2} (2m \cdot 2n)^{\frac{3}{2}} k^{\frac{3}{2}}, \quad 2n_x \cdot 2n_y = 2 \cdot 2N, \]  

   where \( m \) and \( n \) characterize the mode, while \( k = 1, 2, 3, \ldots \) is the degree of excitation.

3. Equation (2) shows that the more complicated the mode is, the higher its spectrum starts, meaning complex modes with large \( m \) and \( n \) are “squeezed” up to higher energies. Since nonclosed—truly stochastic—trajectories can be thought of as the limit of closed trajectories, with \( m \) and \( n \) tending to infinity (by analogy with irrational numbers seen as the limit of a rational \( \frac{m}{n} \) with relatively prime \( m \) and \( n \) tending to infinity), one can say, rather loosely, that quantization squeezes the stochasticity of the system out to infinite energies.

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4. Solving this problem according to wave mechanics, also approximate, has confirmed these conclusions and strengthened them. This demonstrated that the class of symmetric closed orbits considered before according to Bohr and Sommerfeld, exhausts, with one exception to be discussed later, all stationary states of the system. The formula

\[ E_{n_x,n_y} = \left( \frac{3\pi}{16\sqrt{2}} \right)^{\frac{3}{2}} \hbar^{\frac{3}{2}} [(2n_x + 1)(2n_y + 1)]^{\frac{3}{2}}, \quad (2n_x + 1)(2n_y + 1) = (2M + 1), \]  

(3)

where

\[ n_x, n_y = 0, 1, 2, 3 \ldots, \]

which was obtained for the energy, was the same except for the usual addition of the halves.

We can see that both the spectrum found according to Bohr and Sommerfeld and the one found by solving the Schrödinger equation violate the first property in two aspects:

First, the energy values remain unchanged by the exchange \( n_x \leftrightarrow n_y \). This degeneration is unavoidable. It is caused by the fact that although energy is the only integral of motion of the system, in classical mechanics, the quantum description allows new conserved observables—the operators of discrete symmetries, in this case—of the group \( C_{4v} \).

Second, the energy in (2) turns out to be the same for \((n_x, n_y)\) and for \((n'_x, n'_y)\) if

\[ n_x n_y = n'_x n'_y, \]  

(4)

This is presumably an artifact of the approximation, especially since (3) leads to a somewhat different condition,

\[ \left( n_x + \frac{1}{2} \right) \left( n_y + \frac{1}{2} \right) = \left( n'_x + \frac{1}{2} \right) \left( n'_y + \frac{1}{2} \right). \]  

(5)

However, both the emergence of such conditions for the coincidence of energy levels and their close semblance lead one to believe that the exact energies of pairs bounded by (4) or (5) ought to be close.

5. As the multiplicity of an energy level (in the approximation used) or of a group of close energy levels (in the exact solution) is determined by the number of ways in which a natural number \( N \) can be represented by a product of two natural numbers \( n_x \) and \( n_y \) (or the number of ways in which the odd number \( 2M + 1 \) can be represented by a product of two odd factors \( 2n_x + 1 \) and \( 2n_y + 1 \)), it is evident that the value \( E \simeq N^{\frac{3}{2}} \) cannot smoothly depend on the ordinal number \( \nu \) of the state. Thus, the second property of the spectrum of stochastic systems discussed in the literature is confirmed by the calculations in our example. Plots 1 and 2 in Fig. 1 show the dependence of the numbers \( 4N \) or \( 2M + 1 \), which determine the energy levels, on the ordinal number \( \nu \) of the state for quantization according to Bohr and Sommerfeld and quantization by the Schrödinger equation.

6. It is of interest, however, to look for the true cause of this peculiarity. Let us consider, for this purpose, an explicitly integrable system with two degrees of freedom: a square potential well with infinitely high walls. Its total energy \( E \) is the sum of the partial energies of motion in the \( x \) and \( y \) directions.

\[ E = E_x + E_y, \]  

(6)

each being proportional to the square of the corresponding quantum number

\[ E_x \simeq n_x^2, \quad E_y \simeq n_y^2, \quad E \simeq P = n_x^2 + n_y^2. \]  

(7)

Again, we encounter the case where the multiplicity of the level depends on a number theory characteristic, here, the number of ways in which a number \( P \) can be represented by the sum of two squares. Therefore, the dependence of the energy on one quantum number \( \nu \), again, turns out to be irregular (see Plot 3 in Fig. 1) and only "on the average" forms a smooth curve. Now, this is an exact result obtained without any approximations.