Spline Interpolation at Knot Averages

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Abstract. It is well known that when interpolation points coincide with knots, the knot sequence must obey some restriction in order to guarantee the existence and boundedness of the interpolation projector. But, when the interpolation points are chosen to be the knot averages, the corresponding quadratic or cubic spline interpolation projectors are bounded independently of the knot sequence. Based on this fact, de Boor in 1975 made a conjecture that interpolation by splines of order k at knot averages is bounded for any k. In this paper we disprove de Boor's conjecture for k ≥ 20.

We consider infinite spline interpolation. Here is the set-up: suppose that k is a positive integer, and t := (t_j)_{j=-\infty}^{\infty} is a bi-infinite nondecreasing sequence with t_j < t_{j+k}, all j. The jth (normalized) B-spline of order k for the knot sequence t is denoted by N_{j,k}, that is,

\[ N_{j,k}(x) := (t_{j+k} - t_j)[t_j, \ldots, t_{j+k}](x)^{k-1}, \quad \text{all } x \in \mathbb{R}. \]

Let

\[ a := \lim_{j \to -\infty} t_j, \quad b := \lim_{j \to \infty} t_j. \]

Denote by F the linear space of all bounded functions on (a, b) equipped with the sup norm. Let S be the linear span of N_{j,k}, j ∈ Z. Then S ∩ F is the space of bounded polynomial splines of order k at the knot sequence t. Let \( \tau = (\tau_j)_{j=-\infty}^{\infty} \) be an increasing sequence with all \( \tau_j \) in (a, b). Given \( f \in F \), a spline \( s \) is said to interpolate \( f \) at the sequence \( \tau \), if

\[ s(\tau_j) = f(\tau_j), \quad \text{all } j. \]

If there exists a unique bounded operator \( P_k \) from F to \( S \cap F \) such that \( P_k f \) interpolates f for any \( f \in F \), then we say that the bounded interpolation problem is correct. It is shown in [1] that bounding \( P_k \) is equivalent to bounding the inverse of the matrix

\[ G_k := (N_{n,k}(\tau_m))_{n,m=-\infty}^{\infty}. \]
Thus the bounded interpolation problem is correct if and only if the matrix $G_k$ is boundedly invertible.

When the interpolation points are chosen to coincide with the knots, one must impose some restriction on the knot sequence so that the bounded interpolation problem is correct (see [7]). It is not necessary, however, to fix the interpolation points on the knots. One would wonder if we could choose the interpolation points "cleverly" so that any restriction on the knot sequence might be removed. This does succeed in the case when $k = 3$ or $k = 4$, if we choose the interpolation points to be the knot averages:

\[ \tau_m = (t_{m+1} + \cdots + t_{m+k-1})/(k-1). \]

For this choice of interpolation points, Marsden [5] showed

\[ \|G_k^{-1}\| \leq 2, \]

and de Boor obtained (see [2])

\[ \|G_k^{-1}\| \leq 27. \]

This led de Boor to make the following conjecture.

**Conjecture (see [2]).** \[ \|G_k^{-1}\| \leq \text{const}_k < \infty. \]

Such a problem is usually hard to attack. A natural way of mathematical thinking is to consider some simpler cases first. With respect to spline interpolation, the geometric mesh is a good test case. It turns out that de Boor's conjecture is not true for $k \geq 20$. Here is our main result:

**Theorem 1.** Let $(t_m)^{\infty}_{m=-\infty}$ be a geometric mesh: $t_m = q^m$ for some $q > 1$, $\tau_m$ given by (1), and $G_k(q)$ the $\mathbb{Z} \times \mathbb{Z}$ matrix given by

\[ G_k(q)(m, n) = N_{m,k}(\tau_m), \quad m, n \in \mathbb{Z}. \]

Then, for $k \geq 20$, there exists $q(k) > 1$ such that $G_k(q)$, as a mapping from $l_{\infty}$ to $l_{\infty}$, is not invertible.

**Proof.** It is easy to verify that

\[ N_{n+1,k}(qx) = N_{n,k}(x); \]

in particular,

\[ N_{n+1,k}(\tau_{m+1}) = N_{n,k}(\tau_m). \]

Thus $G_k$ is a Toeplitz matrix. In addition, $G_k$ is totally positive (see Chapter 10 of [4]). Therefore $G_k$ is invertible if and only if

\[ \Omega_k(q) := \sum_{n=-\infty}^{\infty} (-1)^n N_{n,k}(\tau_n) \neq 0 \]

(see Chapter 8 of [4]) For simplicity, let

\[ r := \log q, \]

\[ N_k := N_{0,k} \quad \text{and} \quad M_k(x) := N_k(e^{rx}). \]