SOLITARY WAVE TRAINS IN A COLD PLASMA

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The existence of traveling solitary waves, the products of modulation instability in a cold quasi-neutral plasma, is considered. Solitary waves of this type ("solitary wave trains") are formed as a result of bifurcation from a nonzero wave number of the linear wave spectrum. It is shown that the complete system of equations describing the wave process in a cold plasma has solutions of the solitary wave train type, at least when the undisturbed magnetic field is perpendicular to the wave front. Sufficient conditions of existence of solitary wave trains in weakly dispersive media are also formulated.

1. FORMULATION OF THE PROBLEM

We can describe the one-dimensional motions of a cold quasi-neutral plasma by the following dimensionless system of equations [1]:

\[
\frac{\partial n}{\partial t} + \frac{\partial (n u)}{\partial x} = 0
\]

\[
\frac{d u}{d t} + \frac{1}{n} \frac{\partial}{\partial x} \left( \frac{(B_y^2 + B_z^2)}{2} \right) = 0
\]

\[
\frac{d v}{d t} - \frac{1}{n} B \frac{\partial B_y}{\partial x} + \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{1}{n} \frac{\partial B_z}{\partial x} \right) = 0
\]

\[
\frac{d w}{d t} - \frac{1}{n} B \frac{\partial B_z}{\partial x} = \frac{1}{R_i} \frac{\partial}{\partial x} \left( \frac{1}{n} \frac{\partial B_y}{\partial x} \right)
\]

\[
\frac{d B_y}{d t} - B_x \frac{\partial v}{\partial x} + B_y \frac{\partial u}{\partial x} = \frac{1}{R_i} \frac{\partial}{\partial x} \left( \frac{d w}{d t} \right)
\]

\[
\frac{d B_z}{d t} - B_x \frac{\partial w}{\partial x} + B_z \frac{\partial u}{\partial x} = \frac{1}{R_i} \frac{\partial}{\partial x} \left( \frac{d v}{d t} \right)
\]

\[
R_i = \frac{\omega_c}{\omega_0}, \quad R_e = \frac{\omega_c}{\omega_0}, \quad \left( \frac{d}{d t} = \frac{\partial}{\partial x} + u \frac{\partial}{\partial x} \right)
\]

where the independent space variable \( x \), the number density of the particles \( n \), the magnetic induction vector \( B = (B_x, B_y, B_z) \), and the ion velocity \( v = (u, v, w) \) are normalized using the characteristic length \( l \), the number density of the undisturbed plasma \( n_0 \), the undisturbed magnetic field strength vector \( B_0 \), and the Alfvén velocity \( V_A = |B_0|/\left((\pi n_0 (m_i + m_e))^{1/2} \right) \). Here, \( m_i \) and \( m_e \) are the ion and electron masses, \( R_i \) and \( R_e \) are the dispersion parameters, \( \omega_c \) and \( \omega_0 \) are the ion and electron cyclotron frequencies, and \( \omega_0 = V_A / l \) is the characteristic frequency of the phenomenon investigated. For the one-dimensional motions of a cold plasma \( B_z = \text{const} \). At rest the variables \( n, u, v, w, B_x, B_y, \) and \( B_z \) take the values 1, 0, 0, 0, \( \cos \theta \), \( \sin \theta \), and 0, respectively. Here, \( \theta \) is the angle between the magnetic induction vector and the direction of motion of the waves coinciding with the \( x \) axis.

In the case considered the dispersion curve consists of two branches (in Fig. 1 both branches 1 and 2 are reproduced) and for small wave numbers \( k \) (large wave lengths) the phase velocities \( V_{ph}^\star (k) \) have the form:

\[
V_{ph}^\star = 1 - \frac{1}{2R_i R_e} \left\{ 1 - \frac{\sqrt{R_J R_i}}{\sqrt{R_J R_e}} \cos^2 \theta \right\} k^2 + ...
\]

Fig. 1. Dispersion curves.

\[ V_{ph} = \cos \theta \left[ 1 - \frac{1}{2R_e R_i} \left( 1 + \left( \sqrt{R_e} - \sqrt{R_i} \right)^2 \cos^2 \theta \right) k^2 + \ldots \right] \]

Here, \( V_{ph}^+(k) \) and \( V_{ph}^-(k) \) correspond to fast and slow waves, respectively. In the limit as \( k \to 0 \) from (1.2) we obtain \( V_{ph}^+ = V_{ph}^+(0) = 1 \) and \( V_{ph}^- = V_{ph}^-(0) = \cos \theta \).

Solutions of system (1.1) of the traveling wave type depend on \( \xi = x - Vt \) and can be described by a system of equations obtained by simple integration. The integration constants should be so chosen that the state of rest satisfies the equations obtained. The number density of the particles \( n \) and the \( x \) velocity component are connected by algebraic relations with the other unknown functions for which the following differential equations hold:

\[
\frac{1}{n} = 1 - \frac{1}{2V^2} \left( b_y^2 + 2B_y b_y + B_z^2 \right),
\]

\[
u = \frac{1}{V}(b_y^2 + 2B_y b_y + B_z^2), \quad b_y = B_y - B_{y0}, \quad B_{y0} = \sin \theta
\]

\[
\dot{\nu} = b_y \frac{R_e \cos \theta}{V} n \nu, \quad \dot{b}_y = R_e \nu \frac{R_e \cos \theta}{V} n B_z, \quad \dot{B}_z = -R_e \nu \frac{R_e \cos \theta}{V} n b_y
\]

Here, a dot denotes differentiation with respect to \( \xi \).

The possible modes of traveling waves propagating at velocities which differ only slightly from the phase velocities of fast waves of infinite length, i.e., at the velocities \( V = 1 + \mu/2 \), where \( \mu \) is a small parameter, were considered in [2].

In the present paper we will consider questions of the existence of solitary waves with an oscillatory structure of the front in the neighborhood of the point \( k = q \) at which the phase and group velocities coincide (Fig. 1), i.e., \( V = V_0 + \mu \).

\[
V_0 = \sqrt{\frac{\left( (R_e R_i^{-1} + R_i R_e^{-1}) \cos^2 \theta + \sin^2 \theta \right)^2 - 4 \cos^2 \theta}{4 \tan^2 \theta_c \cos^2 \theta}}
\]

\[
\tan \theta_c = \frac{R_e}{R_i} - \frac{R_i}{R_e}, \quad q = \frac{1}{\sqrt{2}} \sqrt{\frac{(R_e^2 + R_i^2) \cos^2 \theta}{V^2} - 2R_e R_i + \frac{R_i R_e \sin^2 \theta}{V^2}}
\]

The velocity \( V_0 \) is equal to the slope of the straight line tangent to the dispersion curve 1 in Fig. 1 at the point \( k = q \), i.e., at the point at which the phase and group velocities of the wave coincide. In formula (1.4) for \( q \) the radicand is positive when \( \theta < \theta_c \). This corresponds to the presence of an inflection point in the dispersion curve 1 in Fig. 1. When \( \theta > \theta_c \) curve 1 is monotonic and there are no points on it at which the phase and group velocities are equal (1:1 resonance points [3]). Solitary waves ("solitary wave trains") can be expected to occur in the neighborhood of the 1:1 resonance points.