ON THE RELIABILITY BOUNDS IN $k$-HNBUE AND $k$-HNWUE CLASSES

CHENG KAN (程侃)
(Institute of Applied Mathematics, Academia Sinica)

HE ZONGFU (何宗福)
(Air Force Engineering College)

HU YIMIN (胡逸民)
(Institute of Applied Mathematics, Academia Sinica)

Abstract

$k$-HNBUE class and its dual were introduced by Basu and Ebrahimi [2]. In this paper we give a generalized definition, and discuss their reliability bounds based on the known mean.

§ 1. Introduction

In reliability theory various classes of life distributions have been introduced to describe aging or wearing out. See, for example, Barlow and Proschan ([1], 1975), Bryson and Siddiqui ([3], 1969), Rolski ([6], 1975), etc. Recently, Basu and Ebrahimi ([2], 1984) proposed a new class of life distributions called $k$-HNBUE ($k$-HNWUE), and studied their properties. But unfortunately, there are some mistakes in [2] due to an incorrect inequality.

In this paper we resume the discussion of the topic. The main results are as follows.

1) We propose a generalized definition for $k$-HNBUE ($0 < k < \infty$) and $k$-HNWUE ($0 < k < \infty$) classes, and examine their properties.

2) The reliability bounds for $k$-HNBUE ($0 < k < \infty$), based on the mean, and the lower bounds are sharp.

3) For $k$-HNWUE ($0 < k < \infty$) similar results are given, and the upper bounds are sharp.

§ 2. Definitions and Simple Properties

Let lifetime $X$ be a nonnegative random variable with distribution function $F(t) \ (\overline{F} = 1 - F)$ and finite mean

$$\mu = \int_0^\infty F'(t) \, dt.$$
\[
\mu(t) = \int_t^\infty F(u) du / F(t).
\] (2.1)

Following the convention, if \( F(t) = 0 \), then \( \mu(t) = 0 \). Obviously, \( \mu(t) \) is the mean residual life.

**Definition 1.** \( \forall k > 0 \), \( F \) (or \( X \)) \( \in k \)-HNBUE if
\[
\frac{1}{\mu(t)} \int_0^t \left[ \frac{1}{\mu(x)} \right]^k dx > 1/\mu, \quad \forall t > 0.
\] (2.2)

If the inequality is reversed in (2.2), then \( F \in k \)-HNWUE.

For simplicity of notations, in the following we denote
\[
B_k = k \)-HNBUE, \quad W_k = k \)-HNWUE, \quad 0 < k < \infty.
\] (2.3)

We list some simple properties.

**Property 1.** \( k = 1 \), \( B_1 = \text{HNBUE}, \ W_1 = \text{HNWUE} \).

Let \( k = 1 \) in (2.2). We have
\[
\frac{1}{\mu(t)} \int_0^t F(x) dx < \exp \left( -t/\mu \right), \quad \forall t > 0.
\] (2.4)

It is the condition for \( F \in \text{HNUE} \).

**Property 2.** If \( X \in B_a(W), \ a > 0 \) constant, then \( aX \in B_a(W) \).

The proof is obvious, and we omit it.

**Property 3.** \( B_k(W) \) is increasing (decreasing) as \( k \) increases.

**Proof.** Using Hölder's inequality
\[
\left\{ \frac{1}{\mu(t)} \int_0^t [g(x)]^i dx \right\} \leq \left\{ \frac{1}{\mu(t)} \int_0^t g(x) dx \right\}^i, \quad 0 < i < 1, \ g(x) > 0,
\]
we have, for \( 0 < k < 1 \),
\[
\frac{1}{\mu(t)} \int_0^t \left[ \frac{1}{\mu(x)} \right]^k dx \leq \left\{ \frac{1}{\mu(t)} \int_0^t \left[ \frac{1}{\mu(x)} \right]^i dx \right\}^{1/k}.
\]

The result follows immediately.

**Property 4.** \( \text{NBUE} \subset B_k, \ \text{NWUE} \subset W_k, \ k > 0 \).

Let
\[
B_0 = \lim_{k \to 0^+} B_k, \quad B_\infty = \lim_{k \to \infty} B_k.
\] (2.5)

\[
W_0 = \lim_{k \to 0^+} W_k, \quad W_\infty = \lim_{k \to \infty} W_k.
\] (2.6)

By property 3,
\[
B_0 = \bigcap_{k > 0} B_k, \quad B_\infty = \bigcup_{k > 0} B_k.
\] (2.7)

\[
W_0 = \bigcap_{k > 0} W_k, \quad W_\infty = \bigcup_{k > 0} W_k.
\] (2.8)

**Theorem 2.1.** \( W_\infty = \text{NWUE} \).

**Proof.** Notice the fact
\[
\lim_{k \to \infty} \left\{ \frac{1}{\mu(t)} \int_0^t \left[ \frac{1}{\mu(x)} \right]^k dx \right\}^{1/k} = \max_{0 < \alpha < \infty} \frac{1}{\nu(x)}, \quad \forall t > 0.
\] (2.9)

Therefore, if \( F \in W_\infty \), we have
\[
1/\mu(t) < 1/\mu, \quad \forall t > 0.
\]

That means \( F \in \text{NWUE} \), which concludes the proof.