Geometry of the $SU_3$ Group

Jerome Epstein

Bloomfield College, Bloomfield, New Jersey 07003

E. L. Schucking

New York University, New York, New York 10003

Received January 6, 1983

A parametrization of the $SU_3$ group is given which is regular in the neighborhood of the unit element. The left-invariant differential forms are explicitly calculated, and a left- and right-invariant metric tensor is exhibited in these parameters.

1. INTRODUCTION

The $SU_3$ group appears in many fields of physics as a symmetry. It was proposed as a symmetry for strong interactions (Gell-Mann, 1962; Ne’eman, 1961). In nuclear physics it was used extensively for the classification of nuclear states (Elliott, 1963). This group gives the canonical transformations which leave the Hamiltonian of the three-dimensional harmonic oscillator invariant (Baker, 1956; Jauch and Hill, 1940). The operations of $SU_3$ transform a solution of the equations of motion of the oscillator into solutions of the same energy. Since the harmonic oscillator is a widely used model in physics, its symmetry group has numerous applications. Quite generally, the group appears always in quantum mechanics when three equivalent states of a physical system have to be considered together, with their possible equivalent descriptions achieved by unitary transformations.

1Work supported in part by a grant from Research Corporation, Providence, Rhode Island.
2. PARAMETRIZATION OF THE MANIFOLD

While even beginners in quantum mechanics know the explicit form of the transformations of the $SU_2$ group, the explicit transformations for $SU_3$ are not so well known. The usual representation of a three-dimensional special-unitary matrix,

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

by means of nine complex (18 real) variables, $u_{ik}$, needs ten side conditions:

$$u_{ij}u_{ik}^* = \delta_{jk}, \quad \det U = 1$$

which are quadratic and cubic in the variables. The group manifold is constructed by the intersection of nine quadratic and one cubic hypersurface in a space of 18 dimensions. Such a representation is not usable for many purposes.

Unitary matrices in any dimension can be parametrized by the exponential map or the Cayley map. The exponential map represents a unitary matrix as

$$U = e^{iH}$$

where $H$ is Hermitian. A special-unitary matrix is obtained with a vanishing trace of $H$ (Chevalley, 1946). Since this is a linear side condition, it can easily be taken into account. For $SU_3$ one has

$$H = \frac{1}{2} \lambda_A x^A \quad (A = 1 \cdots 8)$$

where the $\lambda_A$ are the Gell-Mann matrices, and the $x^A$ are a set of real manifold coordinates. In Appendix A we give the explicit form of the $SU_3$ transformations represented by the exponential map.

Although the general properties of a Lie group can be discussed best in the exponential map, it turns out to be too unwieldy for the calculation of the invariant forms and the metric tensor.

The Cayley map represents a unitary matrix also in terms of a Hermitian matrix, $H$, by

$$U = (1 + iH)(1 - iH)^{-1}$$