The Monadic Second-Order Logic of Graphs, II: Infinite Graphs of Bounded Width*

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Abstract. A countable graph can be considered as the value of a certain infinite expression, represented itself by an infinite tree. We establish that the set of finite or infinite (expression) trees constructed with a finite number of operators, the value of which is a graph satisfying a property expressed in monadic second-order logic, is itself definable in monadic second-order logic. From Rabin's theorem, the emptiness of this set of (expression) trees is decidable. It follows that the monadic second-order theory of an equational graph, or of the set of countable graphs of width less than an integer $m$, is decidable.

Introduction

By a graph we mean a directed hypergraph, the hyperedges of which are labeled by symbols from a finite ranked alphabet, that is equipped with a finite sequence of distinguished vertices.

Finite graphs can be denoted by algebraic expressions (as shown by Bauderon and Courcelle in [3] and [7]) that are constructed over an infinite signature $H$. Infinite graph expressions can be defined, and they denote countable graphs. With finitely many operators, we cannot construct all countable graphs, but only those of finite width. These infinite graph expressions are actually infinite trees. On the other hand, Courcelle has defined in [6], [8], and [10] a monadic second-order logical language appropriate for expressing properties of graphs.

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Our main theorem states that given a finite subset $K$ of $H$, and a monadic second-order formula $\varphi$, we can construct a monadic second-order formula that defines the set $E(K, \varphi)$ of finite or infinite graph expressions constructed over $K$, the corresponding graph of which satisfies $\varphi$. From Rabin's theorem, saying that the monadic second-order theory of the complete binary infinite tree is decidable, it follows that the emptiness of the set $E(K, \varphi)$ is decidable. Hence, the monadic second-order theory of the set of all countable graphs, of width less than some fixed integer, is decidable.

An *equational* graph, i.e., a graph defined as a component of the least solution of a system of graph equations, is the value of a regular (expression) tree. Since a set of trees consisting of a single regular tree is definable in the sense of Rabin, it follows that the monadic second-order theory of an equational graph is decidable.

The paper is organized as follows. Sections 1-4 are devoted to definitions and notations concerning algebraic expressions, infinite trees, and graphs. Section 5 shows how a graph is defined by an infinite graph expression. Section 6 defines monadic second-order logic (as in [8], [10], and [12]). Section 7 gives different characterizations of sets of infinite trees that are definable in monadic second-order logic. Section 8 states the main theorem and Section 9 gives some applications.

1. General Mathematical Notations

We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_+$ the set of positive ones. We denote by $[m, n]$ the interval $\{m, \ldots, n\}$ for $0 \leq m \leq n$ and $[n]$ denotes the interval $[1, n]$ with, in addition, $[0] = \emptyset$.

For sets $A$ and $B$ we denote by $A - B$ the set $\{a \in A / a \notin B\}$. The domain of a partial mapping $f: A \to B$ is denoted by $\text{Dom}(f)$. The restriction of $f$ to a subset $A'$ of $A$ is denoted by $f|A'$. The partial mapping with an empty domain is denoted by $\emptyset$, as the empty set. If two partial mappings $f: A \to B$ and $f': A' \to B$ coincide on $\text{Dom}(f) \cap \text{Dom}(f')$, we denote by $f \cup f'$ their common extension into a partial mapping $A \cup A' \to B$ with domain $\text{Dom}(f) \cup \text{Dom}(f')$.

The cardinality of a set $A$ is denoted by $\text{Card}(A)$. The powerset of $A$ is denoted by $\mathcal{P}(A)$. The set of equivalence relations on $A$ is denoted by $\text{Eq}(A)$.

A binary relation $R$ on a set $A$ is considered as a subset of $A \times A$, so that $xRy$ and $(x, y) \in R$ are equivalent notations. Its transitive closure is denoted by $R^+$, and its reflexive and transitive closure is denoted by $R^*$.

The set of nonempty sequences of elements of a set $A$ is denoted by $A^+$, and sequences are denoted by $(a_1, \ldots, a_n)$ with commas and parentheses. The empty sequence is denoted by $()$, and $A^*$ is $A^+ \cup \{()\}$.

When $A$ is an alphabet, i.e., when its elements are letters, then a sequence $(a_1, \ldots, a_n)$ in $A^+$ can be written unambiguously $a_1a_2 \cdots a_n$. The empty sequence is denoted by $\epsilon$, a special symbol reserved for this purpose. We do the same when $A$ is the set $\mathbb{N}_+$, and $a_1, \ldots, a_n$ are *variables* denoting integers (and not strings of digits). The length of a sequence $\mu$ is denoted by $|\mu|$.