A NUMERICAL REPRESENTATION OF SEMIORDERS
ON A COUNTABLE SET

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Let be a semiorder on a countable set and let if and only if either there exists with or there exists with . Then is a preference relation with transitive indifference, which can be represented by a utility function of the usual sort. It is well known that is represented by a pair of real-valued functions , in the sense that if and only if . We prove that there exists a pair of functions representing , such that is the utility function which represents in the usual sense. Moreover it is easily seen that, for such a pair of functions , we have if and only if either or .

1. Introduction

Fishburn proved that an interval order on a countable set is represented by a pair of real-valued functions , in the sense that if and only if .

In [1], Bridges proved the representation theorem of Fishburn concerning interval orders in a way that is much simpler than the original one. In [2], Bridges showed that it is possible to represent certain interval orders by a single real-valued function. This is accomplished by associating with the interval order a preference relation with transitive indifference, which under suitable conditions may be represented by a utility function of the usual sort. The author pointed out that this is the case of interval orders on a countable set. Bridges observed that, if is an interval order on a countable set , then there exists a pair of functions , representing , such that a preference relation with transitive indifference associated with is represented by the utility function in the usual sense.

In this note we show that these ideas may be used in order to represent a semiorder on a countable set by a single real valued function . Such a function may be easily recovered from a certain representation of the semiorder . We shall prove that, if is a semiorder on a countable set , then there exists a pair of real-
valued functions $u, v$ representing $\succ$, such that a preference relation with transitive indifference associated with $\succ$ is represented by the utility function $u + v$ in the usual sense.

2. Definitions

Let $\succ$ be a binary relation on a set $X$. We say that $\succ$ is a preference relation if

$$x \succ y \Rightarrow \text{not } (y \succ x)$$

(i.e. $\succ$ is asymmetric).

The preference-indifference relation $\succ \sim$ and the indifference relation $\sim$ corresponding to $\succ$ are defined as follows:

$$x \succ \sim y \text{ if and only if } \text{not } (y \succ \sim x),$$
$$x \sim y \text{ if and only if } (x \succ \sim y) \text{ and } (y \succ \sim x).$$

A preference relation $\succ$ which satisfies the following condition:

$$(x \succ x') \text{ and } (y \succ x'') \Rightarrow (x \succ x'') \text{ or } (y \succ x')$$

is said to be an interval order.

Define, for $x, y \in X$,

$$x \succ^* y \text{ if and only if there exists } x' \in X \text{ with } x \succ \sim x' \succ y,$$
$$x \succ^{**} y \text{ if and only if there exists } x'' \in X \text{ with } x \succ \sim x'' \succ \sim y.$$  

In [5], Fishburn showed that, if $\succ$ is an interval order, then $\succ^*$ and $\succ^{**}$ are weak orders (i.e. asymmetric and negatively transitive binary relations). In [2], Bridges proved that $\succ^*$ is a preference relation if and only if $\succ$ is an interval order.

A preference relation $\succ$ which satisfies the following condition:

$$(x \succ x') \text{ and } (x' \succ y) \Rightarrow (x \succ y) \text{ for every } z \in X$$

is said to be a partial semiorder.

Finally, a semiorder is a preference relation that is both an interval order and a partial semiorder.

Now define, for $x, y \in X$,

$$x \succ^0 y \text{ if and only if either } x \succ^* y \text{ or } x \succ^{**} y.$$  

In [5], Fishburn showed that, if $\succ$ is a semiorder, then $\succ^0$ is a weak order.

If $\succ^*, \succ^{**}$ and $\succ^0$ are preference relations, then we shall denote with $\succ \sim_*, \succ \sim^{**}$ and respectively $\succ^0 \sim$ the associated preference-indifference relations.

It is straightforward to prove