An Algebraic Approach to the Planar Coloring Problem

Kauffmann\textsuperscript{1,\,**} and H. Saleur\textsuperscript{2,\,***}

\textsuperscript{1} Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60680, USA
\textsuperscript{2} Department of Physics, Yale University, New Haven, CT 06511, USA

Received June 10, 1992

Abstract. We point out a general relationship between the planar coloring problem with $Q$ colors and the Temperley-Lieb algebra with parameter $\sqrt{Q}$. This allows us to give a complete algebraic reformulation of the four color result, and to give algebraic interpretations of various other aspects of planar colorings.

Introduction

The purpose of this paper is to delineate the relationship of the Temperley-Lieb algebra [TL] with planar graph coloring problems. The main result is a complete algebraic reformulation of the four color theorem [AH]. This reformulation is a special case of a simply stated and more general conjecture about the Temperley-Lieb algebra.

The paper is organized as follows. In the first section we recall the definition of the chromatic polynomial, the dichromatic polynomial, and of the Potts model. In the second section we recall the definition of the Temperley-Lieb algebra and of the Potts model representation. In Sect. Three we prove our first non-trivial result (Proposition 3.1). It states that the Potts model partition function (hence in particular the chromatic polynomial) for any planar graph can be written as the trace of a "transfer matrix" (Definition 3.2), a well defined product of elementary edge operators (Definition 2.3) in the Temperley-Lieb algebra. Such a result was known so far for regular lattices only. In Sect. Four we discuss some properties of the transfer matrices. We show in particular the reciprocal of Proposition 3.1 (Proposition 4.1), namely that an arbitrary product of edge operators can be

\textsuperscript{*} On leave from SphT, Cen Saclay, F-91191 Gif Sur Yvette Cedex, France
\textsuperscript{**} Work supported in part by NSF Grant #DMS-882602, the program for Mathematics and Molecular Biology, UC Berkeley, and a visiting fellowship of the Japan Society for the promotion of science at Kyoto University, Kyoto, Japan
\textsuperscript{***} Work supported in part by DOE Contact #DE-AC02-76ERO3075 and by a Packard Fellowship for Science and Engineering
considered as the transfer matrix for the Potts model on some planar graph. Restricting to the coloring problem we therefore have found that chromatic polynomials of planar graphs occur as traces of fully characterized products of operators in the Potts model representation of the Temperley-Lieb algebra, and conversely the trace of any such product is the chromatic polynomial of some planar graph. The situation compares favorably to the still open problem of characterization of chromatic polynomials [ST]. We also establish some connection between the existence of a transfer matrix that is a product of local edge operators and planarity (Proposition 4.2). Finally, we show how the non-colorability of a graph is expressed by the vanishing of the corresponding transfer matrix. In Sect. Five we consider the algebraic aspects of the planar coloring problem when the number of colors is smaller than four. We show in particular that the non-colorability of a set of simple graphs translates into the vanishing of symmetrizers [Jo, We] in the Potts model representation of the Temperley-Lieb algebra (Proposition 5.3). Section Six deals with the algebraic reformulation of the planar coloring problem. Since the Temperley-Lieb algebra becomes simple for a number of colors greater or equal to four, the approach becomes independent of the Potts model representation and the four color result can be reformulated in a purely algebraic way (Theorem 6.4). In Sect. Seven we present an alternate formulation of these results in terms of the diagrammatic form of the Temperley-Lieb algebra [Ka 1] and a reformulation of the Potts model in terms of link diagrams. This gives alternate proofs for a number of our results, and provides an efficient language for translating between algebra and graph theory. The link diagrammatic approach opens the possibility of using knot theory in the study of these problems. Section Eight contains some speculations about the general properties of real zeroes of chromatic polynomials and the Beraha conjecture [Ba] from the point of view of the Temperley-Lieb algebra. The appendix discusses other (algebraic) reformulations of the four color problem in relation to this work.

While it has sometimes been said [D] that the four color problem is an isolated problem in mathematics, we have found that just the opposite is the case. The four color problem and the generalization discussed here is central to the intersection of algebra, topology and statistical mechanics. We hope that the work presented in this paper will stimulate more investigations of this fascinating structure.

We end this introduction with a brief statement of the Temperley-Lieb algebra, and our general conjecture. The reader will find this material repeated in the text, with appropriate context.

The Temperley-Lieb algebra \((\mathcal{T}L)_n\) is an associative, non-commutative algebra of finite rank over the ring \(\mathbb{R} = \mathbb{Q}[d]\), where \(d\) is an algebraic variable commuting with all elements of \((\mathcal{T}L)_n\). We shall often specialize \(d\) to be a specific real or complex number. \((\mathbb{Q}\) denotes the rational numbers.) The multiplicative generators of \((\mathcal{T}L)_n\) are denoted \(1, e_1, e_2, \ldots, e_{n-1}\), and they satisfy the relations

\[
(a) \quad e_i^2 = de_i, \quad i = 1, \ldots, n - 1, \\
(b) \quad e_i e_{i+1} e_i = e_i, \quad i = 1, \ldots, n - 2, \\
(c) \quad e_i e_j = e_j e_i, \quad \text{if } |i - j| \geq 2.
\]

Thus, a typical element of \((\mathcal{T}L)_3\) is of the form \(a + be_1 + ce_2 + de_1 e_2 + fe_2 e_1\). For example, \((e_1 e_2)(e_1 e_2) = (e_1 e_2 e_1) e_2 = e_1 e_2\) while \((e_1 e_2)(e_2 e_1) = (e_1 e_2 e_2) e_1 = de_1 e_2 e_1\).