Commun. Math. Phys. 156, 527–546 (1993)

Communications in
Mathematical
Physics


# Invariant Connections and Vortices

**Oscar García-Prada**★
Institut des Hautes Etudes Scientifiques, 35 route de Chartres, F-91440 Bures-sur-Yvette, France

Received February 21, 1992; in revised form August 31, 1992

**Abstract.** We study the vortex equations on a line bundle over a compact Kähler manifold. These are a generalization of the classical vortex equations over $\mathbb{R}^2$. We first prove an invariant version of the theorem of Donaldson, Uhlenbeck and Yau relating the existence of a Hermitian–Yang–Mills metric on a holomorphic bundle to the stability of such a bundle. We then show that the vortex equations are a dimensional reduction of the Hermitian–Yang–Mills equation. Using this fact and the theorem above we give a new existence proof for the vortex equations and describe the moduli space of solutions.

## Introduction

In this paper we shall study a direct generalization of the vortex equations on $\mathbb{R}^2$ in which the euclidean plane is replaced by a compact Kähler manifold.

The vortex equations on $\mathbb{R}^2$ were first introduced in 1950 by Ginsburg and Landau [9] in the study of superconductivity. Geometrically they correspond to the equations satisfied by the absolute minima of the Yang–Mills–Higgs functional, defined for a unitary connection $A$ and a smooth section $\phi$ of a Hermitian line bundle over $\mathbb{R}^2$ as

$$\mathrm{YMH}(A, \phi) = \int_{\mathbb{R}^2} |F_A|^2 + |d_A\phi|^2 + \frac{1}{4}(1 - |\phi|^2)^2 .$$

Here $F_A$ is the curvature of $A$ and $d_A\phi$ is the covariant derivative of $\phi$.

If we regard $\mathbb{R}^2$ as the complex plane we may decompose with respect to the complex structure to get $d_A = d'_A + d''_A$. Then by integration by parts we can show that the functional above is bounded below by $2\pi d$, where $d$ is an integer called the *vortex number*, and this minimum is attained if and only if

$$\left.\begin{aligned} d''_A\phi &= 0 \\ F_A &= \frac{1}{2}*(1 - |\phi|^2) \end{aligned}\right\} .$$

★ Current address: Université de Paris-Sud, Mathématique, Bât. 425, F-91405 Orsay Cedex, France

These equations are invariant under gauge transformations and the moduli space of solutions is described by the basic existence theorem of Jaffe and Taubes [13]. They proved that given $d$ points $x_i \in \mathbb{R}^2$ (possibly with multiplicities) there exists a solution to the vortex equations, unique up to gauge equivalence, with $\phi(x_i) = 0$. This means that the moduli space of *vortices* is the space of unordered $d$-tuples, which coincides with the vector space $\mathbb{C}^d$.

The feature of the vortex equations we shall exploit is that they are a *dimensional reduction* of the (anti)-self-dual Yang–Mills equation. More precisely, consider an $SU(2)$ bundle $E$ on a Riemannian 4-manifold $M$. Suppose that $SO(3)$ (or $SU(2)$) acts by isometries on $M$ and that this action can be lifted to $E$. Then $SO(3)$ also acts on the space of connections on $E$, and there is a one-to-one correspondence between $SO(3)$-invariant connections $\mathbf{A}$ and pairs $(A, \phi)$, where $A$ is a unitary connection on a Hermitian line bundle $L$ over the orbit space $M/SO(3)$ and $\phi$ is a section of $L$. The pure Yang–Mills functional of an invariant connection reduces to the Yang–Mills–Higgs functional of $(A, \phi)$. Moreover, $(A, \phi)$ satisfies the vortex equations if and only if the corresponding invariant connection $\mathbf{A}$ satisfies the (anti)-self-dual Yang–Mills equation. In this way, taking $M = \mathbb{R}^2 \times S^2$ Taubes [20] gets the vortex equations over $\mathbb{R}^2$, and taking $M = \mathbb{R}^2_+ \times S^2$ Witten [22] gets the vortex equations over the hyperbolic plane $\mathbb{R}^2_+$.

Taking this invariant point of view we will be able to prove an existence theorem for the more general vortex equations studied in this paper.

In the first section of the paper we introduce these equations. Let $L$ be a Hermitian line bundle over a compact Kähler manifold $X$. If $A$ is a unitary connection on $L$ which is *integrable* (that is, whose curvature has vanishing $(0, 2)$-part), and $\phi$ is a smooth section of $L$, one can define for the pair $(A, \phi)$ a generalized Yang–Mills–Higgs functional depending on a real parameter $\tau$. As in the $\mathbb{R}^2$ case this functional is bounded below by $2\pi\tau d$, where $d$ is the degree of $L$, and this bound is attained if and only if

$$\left.\begin{aligned} &d''_A\phi = 0 \\ &\Lambda F_A - \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau = 0 \end{aligned}\right\},$$

where $\Lambda$ is contraction by the Kähler form. These equations are called the *$\tau$-vortex equations* (though the second equation alone is also sometimes called the *$\tau$-vortex equation*).

It will be convenient to take the equivalent point of view of fixing a holomorphic structure $\bar\partial_L$ on $L$ and fixing a holomorphic section $\phi$ of $\mathscr{L} = (L, \bar\partial_L)$. The $\tau$-vortex equation becomes then the equation

$$\Lambda F_h - \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau = 0$$

for a metric $h$ on $\mathscr{L}$, where $F_h$ is the curvature of the metric connection determined by $\mathscr{L}$ and $h$.

By integrating this equation, we see that a necessary condition for existence of solutions with $\phi \not\equiv 0$ is that

$$\deg L < \frac{\tau \operatorname{Vol} X}{4\pi}.$$