Global Weak Solutions of the Vlasov–Maxwell System with Boundary Conditions*

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Abstract. Boundaries occur naturally in physical systems which satisfy the Vlasov–Maxwell system. Assume perfect conductor boundary conditions for Maxwell, and either specular reflection or partial absorption for Vlasov. Then weak solutions with finite energy exist for all time.

§0. Introduction

We study the initial and boundary value problem of both the non-relativistic and relativistic Vlasov–Maxwell system. We shall prove the global existence of weak solutions under various boundary conditions.

Let $\Omega$ be an open set in $\mathbb{R}^3$ with $C^{1,\mu}$ boundary, for some $\mu > 0$. Consider the non-relativistic Vlasov–Maxwell system:

\[
\begin{cases}
\partial_t f_\beta + \frac{v}{m_\beta} \cdot \nabla_x f_\beta + \frac{e_\beta}{m_\beta} \left( E + \frac{1}{c} v \times B \right) \cdot \nabla_v f_\beta = 0, & 1 \leq \beta \leq N \\
\partial_t E - c \text{curl} B = -j = -4\pi \sum_{\beta} e_\beta \int_{\mathbb{R}^3} f_\beta \, dv, \\
\partial_t B + c \text{curl} E = 0,
\end{cases}
\]  

(VM)

where $0 < t < \infty$, $x \in \Omega$ and $v \in \mathbb{R}^3$, with the constraints

\[
\begin{cases}
\text{div} E = \rho = 4\pi \sum_{\beta} e_\beta \int_{\mathbb{R}^3} f_\beta \, dv, \\
\text{div} B = 0.
\end{cases}
\]  

(0.1)

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The initial conditions are
\[
\begin{aligned}
\{ f_\beta(0, x, v) &= f_{0\beta}(x, v) \quad \text{for } 1 \leq \beta \leq N, \\
\text{div } E_0 &= \rho_0 \quad \text{and} \quad \text{div } B_0 = 0.
\end{aligned}
\] (0.2)

The boundary conditions are
\[
\begin{aligned}
\{ E \times \vec{n} &= 0, \\
f_\beta(t, x, v) &= a_\beta(t, x, v)(Kf_\beta(t, x, v)) + g_\beta(t, x, v), \quad 1 \leq \beta \leq N,
\end{aligned}
\] (0.3)

for \( x \in \partial \Omega \) and \( n \cdot v < 0 \), where \( n \) is the outward normal vector of \( \partial \Omega \) at \( x \). Here the reflection operator is defined as
\[
Kf(t, x, v) = f(t, x, v - 2(v \cdot \vec{n}) \vec{n}),
\] (0.4)

where \( \vec{v} - 2(v \cdot \vec{n}) \vec{n} \) is the reflected vector of \( \vec{v} \) respect to \( \vec{n} \). Also \( N \) is the number of different types of particles with charges \( e_\beta \) and masses \( m_\beta \), \( c \) is the speed of light. The absorption coefficient \( a_\beta(t, x, v) \) and the boundary source \( g_\beta(t, x, v) \) are two given functions on \( n \cdot v < 0 \) satisfying either one of the following conditions:

1. Purely specular reflection condition:
\[
a_\beta(t, x, v) = 1, \quad g_\beta(t, x, v) = 0.
\] (0.5)

2. Partially absorbing condition:
\[
0 \leq a_\beta(t, x, v) \leq a_0 < 1, \quad g_\beta(t, x, v) \geq 0,
\] (0.6)

where \( a_0 \) is a constant. The purely absorbing condition is \( a_\beta \equiv 0 \) and \( g_\beta \equiv 0 \).

These are two typical kinds of the boundary conditions for transport equations. The assumed condition \( E \times \vec{n} = 0 \) comes naturally from physics when \( \Omega \) is surrounded by a perfect conductor. The integrated energy for the non-relativistic case is
\[
\mathcal{E}_T = 4\pi \sum_{\beta}(1 + |v|^2) m_\beta f_\beta dt \, dx \, dv + \int_{(0, T) \times \Omega} (E^2 + B^2) \, dt \, dx.
\] (0.7)

Let \( \chi_T(\cdot) \) be the characteristic function of \([0, T]\). Our main results are as follows.

\textbf{Theorem 0.1 (Non-relativistic case).} Let \( \partial \Omega \in C^{1, \mu} \), for some \( \mu > 0 \). Let \( f_{0\beta} \geq 0 \) a.e., for \( 1 \leq \beta \leq N \), and let \( E_0 \) and \( B_0 \in L^2(\Omega) \) satisfy \( \text{div } E_0 = \rho_0 \) and \( \text{div } B_0 = 0 \) in the sense of distributions. Assume \( f_{0\beta}(1 + |v|^2) \in L^1 \). In the purely specular case (0.5), assume \( f_\beta \in L^\infty \cap L^1 \). In the partially absorbing case (0.6), assume \( f_{0\beta} \in L^p \), \( \chi_T g_\beta(1 + |v|^2) \in L^1 \), for some \( 2 \leq p \leq \infty \) and all \( T < \infty \). Then there exist a weak solution of (VM) in \( 0 < t < \infty, x \in \Omega, v \in \mathbb{R}^3 \) with finite energy \( \mathcal{E}_T \), for all \( T < \infty \). Moreover, if \( f_{0\beta} \in L^q \), \( \chi_T g_\beta \in L^q \), for all \( T < \infty \), then \( \chi_T f_\beta \in L^q \), for all \( T < \infty \), where \( 2 \leq q \leq \infty \).