Limit Behavior of Saturated Approximations of Nonlinear Schrödinger Equation

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Abstract. We consider the solution $u_\varepsilon(t)$ of the saturated nonlinear Schrödinger equation

$$i \frac{\partial u}{\partial t} = -\Delta u - |u|^{4/N} u + \varepsilon |u|^{q-1} u \quad \text{and} \quad u(0, \cdot) = \varphi(\cdot), \quad (1)_\varepsilon$$

where $N \geq 2$, $\varepsilon > 0$, $1 + 4/N < q < (N+2)/(N-2)$, $u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $\varphi$ is a radially symmetric function in $H^1(\mathbb{R}^N)$. We assume that the solution of the limit equation is not globally defined in time. There is a $T > 0$ such that $\lim_{t \to T} \|u(t)\|_{H^1} = +\infty$, where $u(t)$ is the solution of

$$i \frac{\partial u}{\partial t} = -\Delta u - |u|^{4/N} u \quad \text{and} \quad u(0, \cdot) = \varphi(\cdot). \quad (1)$$

For $\varepsilon > 0$ fixed, $u_\varepsilon(t)$ is defined for all time. We are interested in the limit behavior as $\varepsilon \to 0$ of $u_\varepsilon(t)$ for $t \geq T$. In the case where there is no loss of mass in $u_\varepsilon$ at infinity in a sense to be made precise, we describe the behavior of $u_\varepsilon$ as $\varepsilon$ goes to zero and we derive an existence result for a solution of (1) after the blow-up time $T$ in a certain sense. Nonlinear Schrödinger equation with supercritical exponents are also considered.

I. Introduction

In the present paper, we consider the saturated nonlinear Schrödinger equation:

$$i \frac{\partial u}{\partial t} = -\Delta u - |u|^{4/N} u + \varepsilon |u|^{q-1} u \quad \text{and} \quad u(0, \cdot) = \varphi(\cdot), \quad (1)_\varepsilon$$

where $\Delta$ is the Laplace operator on $\mathbb{R}^N$, $u : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{C}$, and $\varphi \in H^1(\mathbb{R}^N)$. We assume that $N \geq 2$, $\varepsilon > 0$ and $1 + 4/N < q < (N+2)/(N-2)$. 
We say that $u(.)$ is a solution of an equation of the type
\[ i\partial u/\partial t = -\Delta u - f(u) \quad \text{and} \quad u(0,.) = \varphi(.) , \]
for $t \in [0, T)$, where $f(x)$ is a nonlinear term, if $\forall t \in [0, T)$,
\[ u(t) = S(t)\varphi + i \int_0^t S(t-s)\{f(u(s))\} \, ds , \]
where $S(.)$ is the group with the infinitesimal generator $i\Delta$ (the free Schrödinger group) and for each $t$, $u(t)$ denotes the function $x \mapsto u(t, x)$.

For a fixed $\varepsilon > 0$, under these assumptions on $N$ and $q$, it is well known that equation (1)$_\varepsilon$ has a unique solution $u_\varepsilon(t)$ in $H^1$ defined globally in time: that is $\forall t \in \mathbb{R}$, $u_\varepsilon(t) \in H^1 = H^1(\mathbb{R}^N)$ (see Ginibre and Velo [5, 6], Kato [8]). The problem is to understand for a fixed $t > 0$, the limit behavior of $u_\varepsilon(t)$ as $\varepsilon$ goes to zero.

Indeed, for both numerical and theoretical reasons, we want to relate this limit behavior to the limit equation:
\[ i\partial u/\partial t = -\Delta u - |u|^{4/N} u \quad \text{and} \quad u(0,.) = \varphi(.) . \] (1)

It is well known that Eq. (1) has a unique solution $u(t)$ in $H^1$ and there exists $T > 0$ such that $\forall t \in [0, T)$, $u(t) \in H^1$ and either $T = +\infty$ or $\lim_{t \to T} \|u(t)\|_{H^1} = +\infty$ (see Ginibre and Velo [5, 6], Kato [8]). In [15], it is shown that as $\varepsilon \to 0$, $u_\varepsilon(t)$ converge in $H^1$ to $u(t)$ uniformly in time in $H^1(\mathbb{R}^N)$ on compact sets of $[0, T)$. Let us consider initial data $\varphi$ such that $T < +\infty$. Moreover, for a fixed $\varepsilon > 0$, $u_\varepsilon(t)$ is globally defined in time. In order to simplify calculations (numerical computations) as well as for physical reasons, the problem is, for fixed $t$, to relate the behavior of $u_\varepsilon(t)$ as $\varepsilon \to 0$ to the nonlinear Schrödinger equation (1). For example, if we can define $\lim_{\varepsilon \to 0} u_\varepsilon(t)$ for $t > T$, in what sense does it satisfy the nonlinear Schrödinger equation with the nonlinear term $-|u|^{4/N}u$?

The other way to see this problem is to see it as a problem of physical continuation of blow-up solutions of the nonlinear Schrödinger equation. Equation (1) appears as a model in a lot of different fields: in nonrelativistic quantum mechanics, in superconductivity, in plasmas, in laser beam propagation ($N = 2$). In particular, for $N = 2$ Eq. (1) can be considered to first approximation as a model of a planar laser beam which is propagating along a single direction $t$ in $\mathbb{R}^3$. In a way, the solution $u(t, x_1, x_2)$ measures the intensity of the laser at a point $(t, x_1, x_2)$ and blow-up of the solution is related to the self-focusing of the laser beam. This model does not quite meet the physicist's requirements when a blowing-up in finite time occurs. In fact, the nonlinear term $-|u|^2u$ is the first term of the expansion of the nonlinearity, and this model is valid when the solution is not too large. Since we have $\lim_{t \to T} \|u(t)\|_{L^4} = +\infty$, the approximation is no longer valid at the blow-up time. For this reason, physicists add a corrective term which gives the saturated model of Eq. (1)$_\varepsilon$ with
\[ f(u) = -|u|^2u + \varepsilon|u|^{q-1}u \]
as nonlinear term where $0 < \varepsilon \ll 1$ and $q > 3$. 