The Wulff Construction and Asymptotics of the Finite Cluster Distribution for Two-Dimensional Bernoulli Percolation

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Abstract. We consider two-dimensional Bernoulli percolation at density \( p > p_c \) and establish the following results:

1. The probability, \( P_N(p) \), that the origin is in a finite cluster of size \( N \) obeys

\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} \log P_N(p) = - \frac{\omega(p) \sigma(p)}{\sqrt{P_\infty(p)}},
\]

where \( P_\infty(p) \) is the infinite cluster density, \( \sigma(p) \) is the (zero-angle) surface tension, and \( \omega(p) \) is a quantity which remains positive and finite as \( p \downarrow p_c \). Roughly speaking, \( \omega(p) \sigma(p) \) is the minimum surface energy of a "percolation droplet" of unit area.

2. For all supercritical densities \( p > p_c \), the system obeys a microscopic Wulff construction: Namely, if the origin is conditioned to be in a finite cluster of size \( N \), then with probability tending rapidly to 1 with \( N \), the shape of this cluster—measured on the scale \( \sqrt{N} \)—will be that predicted by the classical Wulff construction. Alternatively, if a system of finite volume, \( N \), is restricted to a "microcanonical ensemble" in which the infinite cluster density is below its usual value, then with probability tending rapidly to 1 with \( N \), the excess sites in finite clusters will form a single large droplet, which—again on the scale \( \sqrt{N} \)—will assume the classical Wulff shape.

1. Introduction

We consider Bernoulli bond percolation on the square lattice in which bonds are independently occupied with density \( p \) and vacant with density \( 1 - p \). This model is known to have a phase transition at density \( p_c = 1/2 \), below which the occupied clusters are finite with probability one (w.p. 1) and above which there is a unique

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infinite cluster w.p. 1. In this paper, we study the finite occupied clusters throughout the high-density or percolating phase, i.e. whenever \( p > p_c \). Specifically, we obtain detailed estimates on the distribution of sizes and shapes of asymptotically large finite clusters.

In order to motivate our questions and our results, it is worth noting at the outset that the study of large finite clusters in the high-density phase of percolation has an analogue in other statistical mechanics models. The high-density phase of percolation corresponds to the ordered, and hence low-temperature phase of models such as the Ising magnet; the infinite cluster density is the analogue of the spontaneous magnetization \([FK]\) (see also \([ACCN]\)). Thus, in a distributional sense, the infinite cluster in a percolation configuration corresponds to the collection of excess plus spins in a low-temperature plus-state Ising configuration. Similarly, an anomalously large finite cluster in a percolation configuration corresponds to an anomalously large droplet of minus spins in a plus-state Ising configuration; i.e. the asymptotically large finite clusters may be viewed as “droplets of the wrong phase.” In a more general context, the study of the shapes of these clusters is related to the question of crystal formation in other systems: What are the equilibrium shapes of crystals of one phase immersed in another?

1.A. Previous Results. Let us first discuss the size distribution of large finite clusters in percolation. This is typically described by the so-called finite cluster distribution:

\[
P_N(p) = P_p(|C(0)| = N),
\]

(1.1)

where \( P_p(\cdot) \) denotes Bernoulli measure at density \( p \), and \(|C(0)| \) denotes the size of the occupied cluster of the origin. There has been a good deal of previous work on the large-\( N \) behavior of \( P_N(p) \). It has been known for some time that below threshold \( \mu(p) < 1 \),

\[
e^{-c_1(p)N} \leq P_N(p) \leq e^{-c_2(p)N} \quad (p < p_c)
\]

(1.2)

in all dimensions, with \( c_1(p) \) and \( c_2(p) \) positive, finite, dimension-dependent constants. The lower bound in (1.2) is trivial; the upper bound was originally derived in \([H]\), and then rederived in \([K1]\) and \([AN]\). The behavior above threshold is of a very different form; for \( d \) dimensions, \( P_N(p) \) is expected to satisfy

\[
e^{-c_3(p)N(d-1)/d} \leq P_N(p) \leq e^{-c_4(p)N(d-1)/d} \quad (p > p_g)
\]

(1.3)

with \( c_3(p) \) and \( c_4(p) \) positive, finite, dimension-dependent constants. Both bounds in (1.3) were originally derived only for \( p \) near 1 \([KS]\). The lower bound was later shown to hold for all \( p > p_c \) in \([ADS]\). That the upper bound in (1.3) holds for all \( p > p_c \) was demonstrated for two dimensions in \([K3]\) (see also \([CC2]\)). Still later, in \([CCN]\), an upper bound of the form (1.3) with logarithmic modifications (i.e. with \( c_4(p) \) replaced by \( c_4(p)/\log N \)) was shown to hold in dimensions \( d \geq 3 \) whenever \( p \) is above a value\(^1\) which was conjectured to coincide with the percolation

\(^1\) Very recently, there have been two independent proofs that this value coincides with the half-space percolation threshold ([BGN], [Z])