Universal Teichmüller Space and \( \text{Diff} S^1/S^1 \) *

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Abstract. We point out that the coset space \( \text{Diff} S^1/S^1 \) is a dense complex submanifold of the Universal Teichmüller Space \( S \) of compact Riemann spaces of genus \( g \geq 1 \). A holomorphic map of \( S \) into the infinite dimensional Segal disk \( D_1 \) is constructed. This is the Universal analogue of the map of Teichmüller spaces into the Siegel disk provided by the period matrix. The Kähler potential for the general homogenous metric on \( \text{Diff} S^1/S^1 \) is computed explicitly using the map into \( D_1 \). Some applications to string theory are discussed.

There are many reasons to believe that there is a string theory [1] of quantum gravity. Since classical gravity has a natural formulation in terms of Riemannian geometry, it is reasonable to expect that quantum gravity can be formulated in terms of its complex analogue, Kähler geometry. By combining these two surmises, it is natural to seek a formulation of string theory in terms of Kähler geometry. One approach to this was developed by one of us in collaboration with Bowick and Rajeev [2]. In that approach the basic object of study 1 is the coset space \( \text{Diff} S^1/S^1 \), which was proved to be a homogenous Kähler manifold. It was shown that this manifold has a finite Ricci tensor (a non-trivial fact in infinite dimensions) which gives a natural explanation of the critical dimension 26 of string theory.

Complex geometry also arises in the conventional perturbative string theory although in a completely different way. The \( g \)-loop scattering amplitude of string theory can be expressed as an integral of the square of a holomorphic function on a complex manifold, the Teichmüller space of Riemann surfaces of genus \( g \). The

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1 Throughout this paper \( \text{Diff} S^1 \) will denote the group of orientation preserving diffeomorphisms of the circle
measure of integration can be understood [3] in terms of the map of $T_g$ to the Siegel disk $D(g)$. We will explain this in some more detail later in the paper.

It is interesting to ask how the two approaches are related. The approach based on $\text{Diff}S^1/S^1$, while more abstract, holds the promise of a truly non-perturbative approach to string theory. This is important since there is reason to believe that the perturbative expansion does not converge [4]. It was conjectured initially [2] that $\text{Diff}S^1/S^1$ is a “universal moduli space” for Riemann surfaces. The precise connection between the abstract approach based on $\text{Diff}S^1/S^1$ and the more conventional approach based on Riemann surfaces remains, however, obscure. It was later conjectured by one of us (S.G.R.) that all the Teichmüller spaces could be embedded as Kähler submanifolds of $\text{Diff}S^1/S^1$. But progress in this direction was obstructed by the presence of certain divergences. But since then there has been important progress in this direction from the work of Kirillov, Yuriev [5], Nag [6], and Verjovsky.

Their results imply that $\text{Diff}S^1/S^1$ is a dense complex submanifold of $S$, the space of univalent functions on the unit disk. The reason why this is an exciting result is that $S$ is the “Universal Teichmüller Space” in the theory of Bers. More precisely [7], the Teichmüller space $T_g$ of compact Riemann surfaces of genus $g \geq 1$ can be holomorphically embedded into $S$.

These results open up the exciting possibility of a non-perturbative formalism for closed bosonic string theory. It is a natural conjecture [8] that the string amplitude can be written as an integral over $\text{Diff}S^1/S^1$ of the modulus square of a holomorphic function (which can be expressed in terms of infinite dimensional analogues of $\Theta$ functions). The measure of integration would be determined by a homogenous Kähler metric on $\text{Diff}S^1/S^1$. We will give a more precise statement of this idea at the end of this paper.

In constructing this approach to string theory, a holomorphic map of $\text{Diff}S^1/\text{PSL}(2, \mathbb{R})$ into the infinite dimensional Segal disk [9] is important. It is the analogue of the map of the Teichmüller space to the Siegel upper half plane, (provided by the period matrix) in the perturbative approach. It is also useful to understand the geometry of $\text{Diff}S^1/S^1$ and its various embeddings as explicitly as possible. The original approach to the Kähler geometry of $\text{Diff}S^1/S^1$ was rather abstract and relied heavily on the homogeneity of the space. The work of Kirillov allows us to establish explicitly a holomorphic co-ordinate system on $\text{Diff}S^1/S^1$. In this paper we will calculate the Kähler potential of the most general homogenous metric on $\text{Diff}S^1/S^1$, in this co-ordinate system. Kirillov and Yuriev already obtained the Kähler potential for one parameter family of homogenous metrics. But for that special case the Ricci tensor does not exist. We will generalize their result by finding the potential for the general two-parameter family, including those for which the Ricci tensor does exist.

Even apart from string theory, the geometry of $\text{Diff}S^1/S^1$ is relevant to the study of irreducible representations of the Virasoro algebra, and to conformal field theory. For example Segal has constructed the $c = 1$ projective representation of the Virasoro group $\text{Diff}S^1$ by the embedding method. We believe that all irreducible (projective) representations of $\text{Diff}S^1$ can be obtained in terms of modular functions on the homogenous spaces $\text{Diff}S^1/S^1$ and $\text{Diff}S^1/\text{PSL}(2, \mathbb{R})$. Witten [10] has proposed a similar idea.

Before we end this introduction, we have to warn the reader that any discussion of infinite dimensional manifolds is plagued by certain technical difficulties. For example, the tangent space of $\text{Diff}S^1/S^1$ is not complete in the norm defined by its