GENERALIZED VARIATIONAL PRINCIPLES ON NONLINEAR THEORY OF ELASTICITY WITH FINITE DISPLACEMENTS*

Hsueh Dah-wei (薛大方)
(Peking Institute of Technology, Beijing)
(Received April 17, 1990)

Abstract

Two generalized variational principles on nonlinear theory of elasticity with finite displacements in which the $\sigma_{i\ell}$, $e_{i\ell}$ and $u_i$ are all three kinds of independent functions are suggested in this paper. It is proved that these two generalized variational principles are equivalent to each other if the stress-strain relations are satisfied as constraints. Some special cases, i.e., generalized variational principles on nonlinear theory of elasticity with small deformation, on linear theory with finite deformation and on linear theory with small deformation together with the corresponding equivalent theorems are also obtained. All of them are related to the three kinds of independent variables.

Key words finite displacement, nonlinear theory of elasticity, generalized variational principle, three kinds of independent variables

I. Introduction

At the same time when professor Chen made contributions to variational principles such as to develop the method of Lagrange multipliers for establishing generalized variational principle, to indicate that the Hu-Washizu variational principle is actually the principle only with two kinds of independent variables which is equivalent to the well-known Hellinger-Reissner variational principle, to make progress in incompatible finite element methods and to establish equivalent theorems, he suggested the method of high order Lagrange multiplier[1234] and used this method to establish many variational principles such as the generalized variational principles for nonlinear elasticity with finite displacements[56].

Two generalized variational principles on nonlinear theory of elasticity with finite displacements in which the $\sigma_{i\ell}$, $e_{i\ell}$ and $u_i$ are all three kinds of independent functions are suggested in this paper. It is proved that these two generalized variational principles are equivalent to each other if the stress-strain relations are satisfied as constraints. The difference between

* Dedicated to the Tenth Anniversary and One Hundred Numbers of AMM (III)
generalized variational principles which have known before and the corresponding principles in this paper is that the former can usually be considered as the generalization from the principle of minimum potential energy or from the principle of complementary energy while the latter can be considered as the generalization from the superposition of these two principles mentioned above. The difficulty that it is not easy to eliminate the stress-strain relation as a constraint by Lagrange multiplier is overcome in this paper. It can be believed that the thinking in this paper may be used in other fields to establish the corresponding generalized variational principle.

II The Establishment of Generalized Variational Principles

The following equations and conditions must be satisfied if a nonlinear elastic body is under equilibrium with finite displacements.

The equilibrium equations in the entire volume \( V \) of the body

\[
[(\gamma_{ij} + u_{ij}) \sigma_{ij}]_{,j} + F_{i} = 0
\]  
(2.1)

The relations between strain-displacement relation in \( V \)

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j})
\]  
(2.2)

The stress-strain relations in \( V \)

\[
\frac{\partial A}{\partial \varepsilon_{ij}} = \sigma_{ij}
\]  
(2.3a)

or

\[
\frac{\partial B}{\partial \sigma_{ij}} = \varepsilon_{ij}
\]  
(2.3b)

The boundary conditions on the boundary \( S_u \) where the displacements are given

\[
u_i = \bar{u}_i
\]  
(2.4)

The boundary conditions on the remainder boundary \( S_{\sigma} \) where the tractions are given

\( S = S_u + S_{\sigma} \), where \( S \) is the entire surface of \( V \)

\[
(\gamma_{ij} + u_{ij}) \sigma_{ij} n_i = \bar{p}_i
\]  
(2.5)

where \( \sigma_{ij} \), \( \varepsilon_{ij} \) and \( u_i \) are the stresses, strains and displacements respectively, \( \gamma_{ij} \) is the Kronecker symbol.

\[
\gamma_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

\( F_i, \bar{u}_i \) and \( \bar{p}_i \) are the volume forces, displacements and tractions which are all given respectively, \( A(\varepsilon_{ij}) \) and \( B(\sigma_{ij}) \) are the strain energy density and complementary energy density respectively and all of them have positive definite quadratics

\[
\frac{\partial^2 A}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \delta_{ij} \delta_{kl} \geq 0
\]  
(2.6a)

\[
\frac{\partial^2 B}{\partial \sigma_{ij} \partial \sigma_{kl}} \delta_{ij} \delta_{kl} \geq 0
\]  
(2.6b)