Are There Chaotic Tilings?

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Abstract. We develop a class of examples in the form of tiling dynamical systems for use as toy models in statistical mechanics, to analyze the possible existence of disordered crystals. We give the first such models which are disordered in the sense of having no discrete spectrum.

1. Introduction

Ten years ago, Ruelle published the paper “Do turbulent crystals exist?” [7], in which he suggested the existence of real materials which in thermal equilibrium at low temperature would be quite different microscopically from the usual periodic crystals; the suggested difference would be demonstrated by a diffraction spectrum which was absolutely continuous, even at zero temperature.

Ruelle’s argument was based on a comparison of the usual classical statistical mechanical formalism with a typical dynamical system with $\mathbb{R}^3$ action ($\mathbb{R}^3$ representing spatial translations), but without any detailed consideration of the structural role played by interacting particles in the former.

The present paper is motivated by the same problem, but with a different premise. We have chosen to concentrate on the special features which may be due to the role played by the interacting particles in statistical mechanics, with the aim to determine the qualitative low temperature features of generic classical statistical mechanical models with short range interactions. It is well known [7] that no such model has ever been proven to exhibit an ordered (crystalline) phase; presumably the reason is the difficulty in analyzing such models. To obtain results we first restrict attention to zero temperature, and then we distort the models to that of tiling dynamical systems (defined below), as is sometimes done in analyzing quasicrystals [8]. (Roughly speaking, in a tiling dynamical system the phase space

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consists of the tilings of Euclidean n-space, \( E^n \), by copies of some finite set of shapes called tiles; intuitively, the way in which neighboring pairs of tiles need to fit together in a tiling replaces the short range interaction of mechanics.) It has been proven [4] that, generically, statistical mechanical models are uniquely ergodic with respect to spatial translations at zero temperature, and so we use this as an assumption in our models. In summary, the problem of the qualitative behavior of low temperature matter is here translated into: What is the range of qualitative behavior of uniquely ergodic tiling dynamical systems; in particular, do there exist such systems with absolutely continuous spectrum?

We introduce now some definitions. A tiling is a decomposition of \( E^n \) into a union of "tiles" where:

(a) there is a fixed finite set \( \mathcal{P} \) of "prototiles", which are homeomorphic images of the closed n-ball;
(b) each tile is an isometric copy of some prototile,
(c) the interiors of the tiles do not overlap,
(d) the isometries in (b) are restricted to some fixed subgroup \( G \) of the full isometry group of \( E^n \).

We endow the space \( V(\mathcal{P}) \) (assumed nonempty) of all tilings by some given set \( \mathcal{P} \) of prototiles with a topology. Intuitively, tilings should be close if they differ only slightly inside some large bounded region. A finite set of nonoverlapping tiles will be called a swatch. We define a countable base for the topology on \( V(\mathcal{P}) \), using some countable dense subset \( G' \) of the topological subgroup \( G \) (usually \( \mathbb{Z}^n \) or \( \mathbb{R}^n \)) of the isometry group of \( E^n \), as follows. Given a positive integer \( k \), a set of positive rationals \( \{r_j\}_{j=1}^k \), and a swatch of tiles \( \{g'_j(P'_j)\}_{j=1}^k \) (where \( g'_j \in G', P'_j \in \mathcal{P} \)), we define the open set consisting of all tilings containing a swatch \( \{g'_j(P'_j)\}_{j=1}^k \) such that \( h[g'_j(P'_j), g'_i(P'_i)] < r_j \) for all \( j \leq k \), where \( h \) is the Hausdorff metric on compact sets. We note that the space \( V(\mathcal{P}) \), of tilings from a given prototile set \( \mathcal{P} \), is compact and metrizable, and \( G \) acts continuously on \( V(\mathcal{P}) \) [6].

Now consider any (one-dimensional) subshift \((X, T)\) over a finite alphabet \( \mathcal{A} \), with lattice translation denoted by \( T \). We will need to refer on occasion to the cylinder sets \( C_a = \{x \in X: x_0 = a\}, a \in \mathcal{A} \). Then, given a positive real-valued function \( f \) on the alphabet \( \mathcal{A} \), we associate with this subshift \((X, T)\), which is a discrete dynamical system, the continuous dynamical system \((X_f, T_f)\) defined as follows. \( X_f \) is the subset of all tilings of \( E^n \) by translations of closed intervals \([0, a]\), where \( a \) is in the range of \( f \), and the sequence of intervals \( I_i \), of length \(|I_i|\), is such that any corresponding sequence of letters \( f^{-1}(|I_i|) \in \mathcal{A} \) is in \( X \). \( X_f \) is easily seen to be a closed subset of the space of all tilings by such intervals, and invariant under translations, which are denoted by \( \{T_t: t \in \mathbb{R}\} \). (We use below the obvious equivalence of \((X_f, T_f)\) with the classical construct of a flow under a function.)

We now specialize to the case where \((X, T)\) is the substitution dynamical system determined by the substitution \( \xi \):

\[
\xi(0) = 0101, \quad \xi(1) = 1110.
\]  

We note that \((X, T)\) is uniquely ergodic [3], which easily implies that \((X_f, T_f)\) is also uniquely ergodic for any \( f \).