On a Model for Quantum Friction, II. Fermi’s Golden Rule and Dynamics at Positive Temperature

V. Jakšić¹, C.-A. Pillet²

¹ Institute for Mathematics and its Applications, University of Minnesota, 514 Vincent Hall, 55455-0436 Minneapolis, Minnesota, U.S.A.
² Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

Received: 6 December 1994/in revised form: 30 March 1995

Abstract: We investigate the dynamics of an N-level system linearly coupled to a field of mass-less bosons at positive temperature. Using complex deformation techniques, we develop time-dependent perturbation theory and study spectral properties of the total Hamiltonian. We also calculate the lifetime of resonances to second order in the coupling.

1. Introduction

Let $\mathcal{A}$ be a quantum mechanical N-level system with energy operator $H_A$ on the Hilbert space $\mathcal{H}_A = \mathbb{C}^N$. We denote by $E_1 < E_2 < \cdots < E_M$ the eigenvalues of $H_A$ listed in increasing order. We will colloquially refer to $\mathcal{A}$ as an atom or small system. Even though we formulate our results for the N-level system $\mathcal{A}$ most of them will, in some sense, extend to situations where $\mathcal{H}_A$ is infinite dimensional and $H_A$ unbounded – see Remark 4 at the end of Sect. 2 for more details.

Let $\mathcal{B}$ be an infinite heat bath. In this paper $\mathcal{B}$ will be an infinite free Bose gas at inverse temperature $\beta = 1/kT$, without Bose–Einstein condensate. This system is described (see e.g. [BR, D1, D2, LP]) by a triple $\{\mathcal{H}_B, \Omega_B, H_B\}$, where $\mathcal{H}_B$ is a Hilbert space, $H_B$ a self-adjoint operator on $\mathcal{H}_B$, and $\Omega_B$ a unit vector in $\mathcal{H}_B$.

Let us denote by $\omega(k)$ the energy of a boson with momentum $k \in \mathbb{R}^3$. Then the equilibrium momentum distribution of bosons at inverse temperature $\beta$ is given by Planck’s law,

$$\rho(k) = \frac{1}{\exp(\beta \omega(k)) - 1}.$$  

The space $\mathcal{H}_B$ carries a representation of Weyl’s algebra (CCR),

$$W_B(f) = \exp(i\varphi_B(f)),$$  

where the field operators $\varphi_B(f)$ satisfy, for $(1 + \omega^{-1/2})f \in L^2(\mathbb{R}^3)$, the relation

$$(\Omega_B, W_B(f)\Omega_B) = \exp \left[ -\frac{\|f\|^2}{4} - \frac{1}{2} \int_{\mathbb{R}^3} |f(k)|^2 \rho(k) d^3k \right].$$  

(1.2)
The action of $H_B$ is determined by the formula
\[
\exp(itH_B)W_B(f)\exp(-itH_B) = W_B(\exp(it\omega)f).
\] (1.3)

We are interested in the physically realistic case of mass-less bosons: $\omega(k) = |k|$. Let us suppose that the systems $A$ and $B$, isolated at time $t = 0$, start interacting. One expects the temperature of the small system to change. Since the heat reservoir is an infinite system its temperature will remain constant, and thermal equilibrium is achieved when both systems reach the same temperature $1/\beta$. Roughly speaking, this series of papers is devoted to study this approach to thermal equilibrium.

A representation of CCR satisfying Properties (1.1)–(1.3) is usually constructed using the abstract GNS construction. We prefer to work in an explicit representation due to Araki and Woods [AW]. This representation is central in our approach.

The configuration space of a single boson is $\mathbb{R}^3$ and its energy is $\omega(k) = |k|$ (we will always work in the momentum representation). The single particle Hilbert space is $L^2(\mathbb{R}^3)$. Let $\mathcal{H}_b$ be the symmetric Fock space constructed from $L^2(\mathbb{R}^3)$, and denote by $\Omega_b$ its vacuum. Let $a_b(k)$ and $a_b^*(k)$ be the usual annihilation and creation operators on $\mathcal{H}_b$ (see [RS2] for definitions, note that $a_b^*(f) = \int d^3k a_b^*(k)f(k)$ is linear in $f$, while $a_b(f) = [a_b^*(f)]^*$ is anti-linear). Define the energy operator by
\[
H_b = \int_{\mathbb{R}^3} d^3k \omega(k)a_b^*(k)a_b(k),
\]
and the field operators by
\[
\varphi_b(f) = \frac{1}{\sqrt{2}}(a_b(f) + a_b^*(f)).
\]

In the Araki–Woods representation the triple \{\mathcal{H}_B, \Omega_B, H_B\} is given by
\[
\mathcal{H}_B = \mathcal{H}_b \otimes \mathcal{H}_b, \quad \Omega_B = \Omega_b \otimes \Omega_b, \quad H_B = H_b \otimes I - I \otimes H_b.
\]
The annihilation and creation operators are
\[
a_B(f) = a_b((1 + \rho)^{1/2}f) \otimes I + I \otimes a_b^*(\rho^{1/2}f),
\]
\[
a_B^*(f) = a_b^*((1 + \rho)^{1/2}f) \otimes I + I \otimes a_b(\rho^{1/2}f),
\]
and the field operators are given by
\[
\varphi_B(f) = \frac{1}{\sqrt{2}}(a_B(f) + a_B^*(f)).
\]

Notation. We write $A$ instead of $A \otimes I$ or $I \otimes A$, whenever the meaning is clear within the context.

When the thermal bath is at zero-temperature, the following formalism is used to describe the system $A + B$: The Hilbert space of the system is $\mathcal{H}_A \otimes \mathcal{H}_B$ and its Hamiltonian is given by
\[
\tilde{H}_\lambda = H_A \otimes I + I \otimes H_B + \lambda Q \otimes \varphi_b(\alpha) = H_A + H_B + \lambda \tilde{H}_f.
\] (1.4)

There $Q$ is a self-adjoint operator on $\mathcal{H}_A$, $\alpha \in L^2(\mathbb{R}^3)$ and $\lambda \in \mathbb{R}$. In the sequel we will refer to $\alpha$ as the form factor and $\lambda$ as the friction constant. If $\omega^{-1/2}\alpha \in L^2(\mathbb{R}^3)$,