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Diff(S¹) and the Teichmüller Spaces

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Abstract. Precisely two of the homogeneous spaces that appear as coadjoint orbits of the group of string reparametrizations, Diff(S¹), carry in a natural way the structure of infinite dimensional, holomorphically homogeneous complex analytic Kähler manifolds. These are \( N = \text{Diff}(S^1)/\text{Rot}(S^1) \) and \( M = \text{Diff}(S^1)/\text{Möb}(S^1) \). Note that \( N \) is a holomorphic disc fiber space over \( M \). Now, \( M \) can be naturally considered as embedded in the classical universal Teichmüller space \( T(1) \), simply by noting that a diffeomorphism of \( S^1 \) is a quasisymmetric homeomorphism. \( T(1) \) is itself a homomorphically homogeneous complex Banach manifold. We prove in the first part of the paper that the inclusion of \( M \) in \( T(1) \) is complex analytic.

In the latter portion of this paper it is shown that the unique homogeneous Kähler metric carried by \( M = \text{Diff}(S^1)/\text{SL}(2, \mathbb{R}) \) induces precisely the Weil–Petersson metric on the Teichmüller space. This is via our identification of \( M \) as a holomorphic submanifold of universal Teichmüller space. Now recall that every Teichmüller space \( T(G) \) of finite or infinite dimension is contained canonically and holomorphically within \( T(1) \). Our computations allow us also to prove that every \( T(G) \), \( G \) any infinite Fuchsian group, projects out of \( M \) transversely. This last assertion is related to the “fractal” nature of \( G \)-invariant quasicircles, and to Mostow rigidity on the line.

Our results thus connect the loop space approach to bosonic string theory with the sum-over-moduli (Polyakov path integral) approach.

Introduction

Part I: The Complex Structures. The group Diff(S¹) and its universal central extension, the Virasoro group, occurs in string theory as the space of reparametrizations of a closed string. Two coadjoint orbit spaces of Diff(S¹), namely, \( N = \text{Diff}(S^1)/S^1 \) and \( M = \text{Diff}(S^1)/\text{SL}(2, \mathbb{R}) \), have occurred in the physics literature

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¹ The Virasoro group (see Witten [16])
as critically important because precisely these two carry the structure of infinite dimensional, holomorphically homogeneous, complex (Kähler) manifolds (Witten [16]). The complex structures on these spaces are obtained by placing a natural physics-motivated almost complex structure (arising from Kirillov–Kostant representation theory) on the appropriate spaces of real vector fields on $S^1$. This complex structure, arising from “conjugation” of Fourier series, appears, for example, in Pressley [18].

Now, considering diffeomorphisms of $S^1$ as quasisymmetric homeomorphisms one can naturally identify $M$ as embedded in the classical universal Teichmüller space $T(1)$. $T(1)$ is a holomorphically homogeneous complex Banach domain from the famous Ahlfors–Bers theory of the Teichmüller spaces. Our first main result is that this inclusion of $M$ into $T(1)$ is complex analytic. In fact, $M$ is one leaf of a holomorphic foliation of $T(1)$. Also, since $N$ is a holomorphic disc fiber space over $M$, it seems to us that this naturally and directly connects the string reparametrization complex manifolds with the complex analytic moduli of Riemann surfaces. It appears to have been an important question (see, for example, Bowick [5], Bowick and Rajeev [6]) to relate these reparametrization spaces with the spaces of moduli of Riemann surfaces because that would connect the loop space (“geometrical quantization”) approach to string theory with the path integral (“sum over moduli”) approach. Indeed, $T(1)$ contains canonically within itself as complex submanifolds all the Teichmüller spaces of arbitrary Riemann surfaces or Fuchsian groups. If, therefore, strings are reparametrized using the more general quasisymmetric homeomorphisms of the circle (rather than only by smooth diffeomorphisms), then the corresponding $SL(2, \mathbb{R})$ orbit space is the universal Teichmüller space of Riemann surfaces.

Our method of proof is to show that the almost complex structure obtained by the physicists (Bowick and Rajeev [6, 7]; Bowick and Lahiri [8]) on real vector fields on $S^1$ modulo the Möbius vector fields coincides with the almost complex structure of $T(1)$ at the origin. The holomorphic homogeneity of both $M$ and $T(1)$ under the action of (right-) translation then implies that the complex structures are compatible everywhere.

**Part II: The Kähler Structures.** The infinite dimensional holomorphically homogeneous (Fréchet) complex manifold $M = \text{Diff}(S^1)/SL(2, \mathbb{R})$ is shown in Part I to be naturally embedded in universal Teichmüller space, $T(1)$, as one leaf of a holomorphic foliation of $T(1)$. That result showed that these two “universal moduli spaces” are intimately related. There is a unique (up to a scaling factor) homogeneous Kähler metric, $g$, on $M$; this metric and its curvature have been studied intensively by many physicists including Bowick, Rajeev, Lahiri, Zumino, Kirillov (see [5–8, 12, 13]). The chief result now is that this canonical metric $g$ produces precisely the Weil–Petersson Kähler metric on the Teichmüller spaces.²

Let us be more precise. The metric $g$ assigns a hermitian inner product on smooth real vector fields on the unit circle $S^1$. ($S^1$ is to be thought of as the

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² The Kähler form of the metric $g$ is is precisely the symplectic form that $M$ carries by virtue of being a coadjoint orbit manifold. See Witten [16] and Kirillov [19]