On the Spectra of Schrödinger Operators

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Abstract: We give two formulas for the lowest point $\mathcal{J}$ in the spectrum of the Schrödinger operator $L = -(d/dt)p(d/dt) + q$, where the coefficients $p$ and $q$ are real-valued, bounded, uniformly continuous functions on the real line. We determine whether or not $\mathcal{J}$ is an eigenvalue for $L$ in terms of a set of probability measures on the maximal ideal space of the $C^*$-algebra generated by the translations of $p$ and $q$.

Introduction

In this paper, we will study the Schrödinger operator

$$L = -\left(\frac{d}{dt}\right)p\left(\frac{d}{dt}\right) + q$$

on $\mathcal{D} \subset L^2(\mathbb{R})$. As usual, the domain $\mathcal{D}$ of this operator is the collection of functions $f \in L^2(\mathbb{R})$ which have the property that $f$ and $f'$ are absolutely continuous functions on every finite interval and $f'$, $f'' \in L^2(\mathbb{R})$. We assume that $p$ and $q$ are real-valued, bounded, uniformly continuous functions on $\mathbb{R}$. In addition, we assume that $p'$ is also a bounded, uniformly continuous function on $\mathbb{R}$ and that there is a $c > 0$ such that $p(t) \geq c$ for every $t \in \mathbb{R}$. It is well known that, under these assumptions, $L$ is a self-adjoint operator on $\mathcal{D}$. The main goal of this paper is to study the lowest point $\mathcal{J} = \inf \{\lambda : \lambda \in \sigma(L)\}$ of the spectrum of $L$. There have been estimates of the value $\mathcal{J}$ in the literature when the coefficients $p$ and $q$ of the operator have recurrence properties [4]. We will give two formulas for the value $\mathcal{J}$. These formulas are related to a $C^*$-algebra associated with the functions $p$ and $q$.

Before we state our results, some definitions are necessary. For a function $f$ defined on $\mathbb{R}$, by a translation of $f$ we mean a function $f_s$ given by the formula $f_s(t) = f(t+s)$. We denote by $\mathcal{A}$ the $C^*$-algebra generated by all the translations of $p$, $p'$, $q$ and all

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the constant functions on \( \mathbb{R} \). Let \( \mathcal{A}^1 = \{ f \in \mathcal{A} : f' \in \mathcal{A} \} \). For each state \( \varphi \) on \( \mathcal{A} \), let \( H_\varphi \) be the Hilbert space completion of \( \mathcal{A} \) with respect to the inner product \( \langle f, g \rangle_\varphi = \varphi(fg) \). Let \( \mathcal{M} \) be the collection of states \( \varphi \) on the \( C^* \)-algebra \( \mathcal{A} \) such that 
\[
|\varphi(f')| \leq C_\varphi \sqrt{\varphi(f^2)} \]
for every \( f \in \mathcal{A}^1 \), where \( C_\varphi \) is a constant depending only on \( \varphi \). Equivalently, \( \mathcal{M} \) is the collection of states \( \varphi \) for which there is a unique \( h_\varphi \in H_\varphi \) such that 
\[
\varphi(f') = \langle f, h_\varphi \rangle_\varphi \]
for every \( f \in \mathcal{A}^1 \). Let
\[
\mathcal{F}_0 = \{ pw^2 + (pw)' : w \text{ is any real-valued function in } \mathcal{A}^1 \},
\]
and let \( \mathcal{F} \) be the closure of the convex hull of \( \mathcal{F}_0 \) in the norm topology. Let 
\[
d_0(\varphi) = \inf\{ ||\varphi - u||_\infty : u \in \mathcal{F}_0 \} \quad \text{and} \quad d(\varphi) = \inf\{ ||\varphi - u||_\infty : u \in \mathcal{F} \}.
\]

**Theorem 1.**

(a) \( d_0(f) = d(f) \) for every \( f \in \mathcal{A} \).
(b) \( \mathcal{F} = ||q||_\infty - d(q) - ||q||_\infty \).
(c) If \( \mathcal{F} \leq 0 \), then \( I = -d(q) \).
(d) \( \mathcal{F} = \min\{ \varphi(q) + \frac{1}{4} \langle ph_\varphi, h_\varphi \rangle_\varphi : \varphi \in \mathcal{M} \} \).
(e) If \( d(q) > 0 \), then \( -\mathcal{F} = d(q) = \max\{ -\varphi(q) - \frac{1}{4} \langle ph_\varphi, h_\varphi \rangle_\varphi : \varphi \in \mathcal{M} \} \).

We particularly emphasize the fact that in (d) and (e) above, the extrema are attainable. We will explain in Sect. 3 that the fact that they are attainable makes \( \mathcal{F} \) a “quasi-eigenvalue” for \( L \). In other words, we assert that when the coefficients \( p \) and \( q \) satisfy our assumptions, the lowest point in the spectrum of \( L = -(d/dt)p(d/dt) + q \) is always a quasi-eigenvalue. In fact quasi-eigenvalue is the most that one can say about \( \mathcal{F} \) in general. Although \( \mathcal{F} \) can be a genuine eigenvalue, in the case \( p \) and \( q \) are almost periodic functions, it is known that \( \mathcal{F} \) is not an eigenvalue in probability 1.

It is obvious that for each \( s \in \mathbb{R} \), the map \( \varphi_s : f \mapsto f_s \) is an automorphism of the \( C^* \)-algebra \( \mathcal{A} \). The fact that the functions in \( \mathcal{A} \) are uniformly continuous on \( \mathbb{R} \) implies that the group of automorphisms \( \{ \varphi_s : s \in \mathbb{R} \} \) is strongly continuous in the sense that for every \( f \in \mathcal{A} \), \( s \mapsto \varphi_s(f) = f_s \) is a continuous map from \( \mathbb{R} \) into \( \mathcal{A} \). Therefore the automorphism group \( \{ \varphi_s : s \in \mathbb{R} \} \) induces a strongly continuous group of homeomorphisms \( \{ \alpha_s : s \in \mathbb{R} \} \) of the maximal ideal space \( \Omega \) of \( \mathcal{A} \). In other words, the map \( (\omega, s) \mapsto \alpha_s(\omega) \) from \( \Omega \times \mathbb{R} \) to \( \Omega \) is continuous. If we identify \( \mathcal{A} \) with \( C(\Omega) \), then obviously \( \mathcal{A} \) can be regarded as the subset \( C^1(\Omega) \) of \( f \in C(\Omega) \) such that the limit \( f' = \lim_{\varepsilon \to 0} (f \circ \alpha_s - f)/\varepsilon \) exists in the norm topology of \( C(\Omega) \). Similarly, \( \mathcal{M} \) can be identified with the collection of probability measures \( \mu \) on \( \Omega \) such that 
\[
\int_{\Omega} f' \, d\mu \leq C_\mu \left[ \int_{\Omega} |f|^2 \, d\mu \right]^{1/2}
\]
for every \( f \in C^1(\Omega) \), where \( C_\mu > 0 \) is a constant which depends only on \( \mu \). Given a \( \mu \in \mathcal{M} \), \( D_\mu : f \mapsto f' \) is a linear operator from the dense subspace \( C^1(\Omega) \) into \( L^2(\Omega, \mu) \). It seems that the subscript of the symbol \( D_\mu \) is unnecessary, for the operator itself is actually independent of the measure \( \mu \). The reason we write \( D_\mu \) is that its adjoint \( D_\mu^* \) does in general depend on the measure \( \mu \). If we let \( \hat{p} \) and \( \hat{q} \) denote the Gelfand transforms of \( p \) and \( q \) respectively, then it follows from Theorem 1 that the set
\[
\mathcal{M}(p, q) = \left\{ \mu \in \mathcal{M} : \mathcal{F} = \langle \hat{q}, 1 \rangle_\mu + \frac{1}{4} \langle \hat{p}D_\mu^* 1, D_\mu^* 1 \rangle_\mu \right\}
\]