SINGLE SERVER WITH LINEAR STATE-DEPENDENT MEAN RATE

J.A. MORRISON

AT&T Bell Laboratories, Murray Hill, New Jersey 07974, U.S.A.

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Abstract

A birth-death queueing system with a single server, first-come first-served discipline, Poisson arrivals and mean service rate which depends linearly on the number of customers in the system, is considered. Explicit expressions are derived for the equilibrium densities of the sojourn and waiting times. Simple approximations to the densities, including the first order correction terms, are obtained in a heavy-traffic situation.

Keywords: Birth-death queueing systems, heavy-traffic approximation, single server queue, sojourn and waiting times, state-dependent service.

1. Introduction

We consider a birth-death queueing system with a single server, first-come first-served discipline, unlimited queue size, Poisson arrivals, with mean rate $\lambda$, and mean service rate

$$\mu_i = \sigma + i,$$  \hspace{1cm} (1.1)

where $\sigma > -1$, when there are $i \geq 1$ customers in the system. For convenience of notation, we have chosen the unit of time so that $\mu_{i+1} - \mu_i = 1$. The equilibrium probabilities of the number in the system are well known (Kleinrock [3]), and the mean sojourn and waiting times may be obtained from these by Little's result.

In this paper we derive explicit expressions for the equilibrium densities of the sojourn and waiting times. In §2 we formulate the problem, starting from the formulation given by Morrison [5] for general state-dependent mean service rate $\mu_i$. However, instead of taking Laplace transforms as we did there, we work directly in the time domain. The formulation leads to a system of partial difference-differential equations. Single generating functions are introduced which satisfy an ordinary difference-differential equation.

In §3 we consider the case when $\sigma = l$, a non-negative integer, and introduce a double generating function which satisfies a first order partial differential equation, which may be solved explicitly. This leads to explicit expressions for the equilibrium density $s(t)$ of the sojourn time, and the continuous part $w(t)$ of the
equilibrium density of the waiting time. The equilibrium probability that a customer does not have to wait for service is equal to the equilibrium probability $P_0$ that the system is empty. In §4 we consider the case when $\sigma > -1$ is not necessarily an integer. We verify directly that the expressions derived for the single generating functions when $\sigma = l$ remain valid in this case. We then obtain explicit expressions for $s(t)$ and $w(t)$, which involve modified Bessel functions with argument $2\lambda [e^{-\lambda(t-1)} + e^{-\lambda}]^{1/2}$.

In §5 we consider the heavy-traffic case in which $\sigma = \lambda + \gamma \sqrt{\lambda}$, where $\lambda \gg 1$ and $\gamma = O(1)$, and $\gamma$ may have either sign. From the explicit expressions for $s(t)$ and $w(t)$ we show that
\begin{equation}
\begin{aligned}
    s\left(\eta/\sqrt{\lambda}\right) &= \lambda P_0 e^{-\gamma \eta} e^{-\eta^2/2} \left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right], \\
    w\left(\eta/\sqrt{\lambda}\right) &= \lambda P_0 e^{-\gamma \eta} e^{-\eta^2/2} \left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right],
\end{aligned}
\end{equation}
for $0 \leq \eta = O(1)$, and we also derive the first order correction terms. The equilibrium probability $P_0$ that the system is empty is obtained from the normalization condition that $s(t)$ is a density.

2. Formulation of the problem

We consider a birth-death queueing system with a single server, first-come first-served discipline and Poisson arrivals, with mean rate $\lambda$. The problem of determining the equilibrium densities of the sojourn and waiting times for general state-dependent mean service rate $\mu_i$ was formulated by Morrison [5]. It was assumed that $\lambda/\mu_i < \kappa < 1$ for all $i \geq i_0$, so that the system is ergodic. The equilibrium probability $P_i$ that there are $i$ customers in the system is (Kleinrock [3])

\begin{equation}
P_i = P_0 \lambda^{i} / \prod_{j=1}^{i} \mu_j, \quad i = 1, 2, \ldots,
\end{equation}

where
\begin{equation}
P_0 \left(1 + \sum_{i=1}^{\infty} \lambda^{i} / \prod_{j=1}^{i} \mu_j\right) = 1.
\end{equation}

Since the arrivals are Poisson, $P_i$ is also the equilibrium probability that an arriving customer finds $i$ customers already in the system.

We consider a tagged customer that arrives when the system is in equilibrium. Any customers that are already in the system when the tagged customers arrives must be served first, and during the sojourn time of the tagged customer other customers may arrive. We say that the system is in state $(m, n)$ if there are $n$