Pure Point Spectrum Under 1-Parameter Perturbations and Instability of Anderson Localization

A. Ya. Gordon
MITPAN, Warshavskoye shosse, 79, korpus 2, Moscow, 113556, Russia

Received: 22 March 1993

Abstract: We consider a selfadjoint operator, \( A \), and a selfadjoint rank-one projection, \( P \), onto a vector, \( \psi \), which is cyclic for \( A \). We study the set of all eigenvalues of the operator \( A_t = A + tP \) (\( t \in \mathbb{R} \)) that belong to its essential spectrum (which does not depend on the parameter \( t \)). We prove that this set is empty for a dense set of values of \( t \). Then we apply this result or its idea to questions of Anderson localization for 1-dimensional Schrödinger operators (discrete and continuous).

1. Introduction

Let \( \{A_t\}_{t \in \mathbb{R}} \) be a one-parameter family of linear selfadjoint operators

\[ A_t = A + tP \]  

in a Hilbert space \( \mathcal{H} \). Here \( A \) is a selfadjoint operator with simple spectrum and \( P \) a projection \( (\cdot, \psi)\psi \), where \( \psi \) is a normed cyclic vector for \( A \). All the operators (1) are selfadjoint on \( D(A) \) and have the same essential spectrum\(^1\); denote this closed subset of \( \mathbb{R} \) by \( \Sigma \).

We will be concerned with the eigenvalues of the operators \( A_t \) that lie on the essential spectrum, i.e., with the intersection \( \sigma_p(A_t) \cap \Sigma \) [in the sequel, \( \sigma_p(B) \), for any linear operator \( B \), will denote its point spectrum]. Information about this set can help in studying the nature of the spectrum of \( A_t \).

In [SW], a necessary and sufficient condition was found for operators \( A_t \) to have pure point spectrum for \( L \)-a.e. \( t \). It was formulated in terms of the Stieltjes transform of the spectral measure of \( \psi \) for \( A \). This criterion was then applied to questions of Anderson localization for random Schrödinger operators. It led, in particular, to the

\(^1\) Recall that the essential spectrum \( \sigma_{\text{ess}}(B) \) of a selfadjoint operator \( B \) (see [Gl]) consists of all nonisolated points of its spectrum and all isolated eigenvalues of infinite multiplicity. The latter case is impossible for operators whose spectrum has finite multiplicity, in particular for the above operators (1) and for one-dimensional difference and differential operators considered in the following.
following statement (see also [DLS]). Let $h(\omega)$ be the random Schrödinger operator of the one-dimensional Anderson model, i.e.

$$
(h(\omega)y)(n) = y(n-1) + y(n+1) + v(n, \omega)y(n) \quad (n \in \mathbb{Z}),
$$

(2)

where $\omega \in \Omega \{ (\Omega, \mathcal{T}, P) \) is a probability space], and values of random potential $v(n, \omega)$ at distinct sites $n$ are independent real random variables with a common distribution $\nu(dx)$. Suppose $\nu$ has a non-trivial absolutely continuous component and $\int (\log^+ |x|) \nu(dx) < \infty$. Then with probability 1, for $L$-a.e. $t \in \mathbb{R}$ (in particular, for $t = 0$) the operator $h_t(\omega) = h(\omega) + t\delta_0$ has only point spectrum. (Here $\delta_0$ denotes multiplication by the $\delta$-function supported at 0.)

A question arises: is the restriction “for $L$-almost every $t$” in the above result essential? In [SW], an example is indicated where the spectra of operators (1) are pure point for Lebesgue almost all rather than all $t$. However, this example does not provide an answer to a question: what is the situation with Schrödinger operators in $l^2(\mathbb{Z})$? Is it true that for a “typical” Schrödinger operator $h_\omega$ (i.e. for any “sufficiently random” stochastic potential, with probability 1) the spectrum of an operator, say, $h(\omega) + t\delta_0$ is pure point for all $t$? Short of that, is it true for the above-mentioned Anderson model, where the “randomness” of a potential is maximal?

Molchanov raised a related question, which pertained to a one-parameter family of self-adjoint Schrödinger operators $H^\vartheta$ in $L^2(\mathbb{R}_+)$ [see formulas (8) and (9) below]. If a bounded stochastic potential $v = v(x, \omega)$ is ergodic and nondeterministic, then, as it was found by Kotani [Ko], with probability 1 the operator $H^\vartheta(\omega)$ has only point spectrum for $L$-a.e. $\vartheta \in [0, \pi)$. Similarly, the spectrum proved to be pure point for some explicit individual potentials (see example [Go] and general result [KMP], which concerns Schrödinger operators in $l^2(\mathbb{Z}_+)$), but again under the same restriction: for $L$-a.e. value of the boundary phase. Molchanov raised a question: is this restriction essential? There are known some random potentials for which almost surely (a.s.) the spectrum is pure point for $L$-a.e. rather than all $\vartheta$. However, these potentials are highly symmetrical. Is it true that for a “typical” individual potential the spectrum is pure point for all $\vartheta$?

The main result of this paper states that the opposite is true: the restriction “for $L$-almost every $\vartheta$” (or $t$) is quite necessary, at least when the essential spectrum $\Sigma$ has a non-empty interior (which does hold for “typical” potentials). First we establish, for abstract operators (1), the following result.

**Theorem 1.** There is a thin set $Z$ in $\mathbb{R}$ such that for any $t \in \mathbb{R} \setminus Z$ the intersection $\sigma_p(A_t) \cap \Sigma$ is empty.

Then we apply this theorem or its idea to one-dimensional Schrödinger operators and obtain some “negative” results, which show that the localization, though being typical, is however extremely unstable under one-parameter perturbations.

These further results are formulated below. In particular, the answers to the above two questions are contained in Theorems 2*, 2**, and 5.

(a) We begin with results concerning operators on $\mathbb{Z}$. Let $\{h_t\}$ be a one-parameter family of linear selfadjoint operators $h_t$ in $l^2(\mathbb{Z})$ defined by

$$
(h_t y)(n) = y(n-1) + y(n+1) + (v(n) + t\varphi(n))y(n), \quad n \in \mathbb{Z}.
$$

(3)

Here $t \in \mathbb{R}$; $v(\cdot)$ (“the potential”) and $t\varphi(\cdot)$ (“the perturbation”) are real-valued functions on $\mathbb{Z}$, $\varphi \neq 0$ being non-negative and finitely supported. We denote its support by $S$, so that $1 \leq |S| < \infty$. The operators $h_t$ are selfadjoint on the linear