On $E_{10}$ and the DDF Construction

R.W. Gebert*, H. Nicolai

IIInd Institute for Theoretical Physics, University of Hamburg
Luruper Chaussee 149, D-22761 Hamburg, Germany

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Abstract: An attempt is made to understand the root spaces of Kac Moody algebras of hyperbolic type, and in particular $E_{10}$, in terms of a DDF construction appropriate to a subcritical compactified bosonic string. While the level-one root spaces can be completely characterized in terms of transversal DDF states (the level-zero elements just span the affine subalgebra), longitudinal DDF states are shown to appear beyond level one. In contrast to previous treatments of such algebras, we find it necessary to make use of a rational extension of the self-dual root lattice as an auxiliary device, and to admit non-summable operators (in the sense of the vertex algebra formalism). We demonstrate the utility of the method by completely analyzing a non-trivial level-two root space, obtaining an explicit and comparatively simple representation for it. We also emphasize the occurrence of several Virasoro algebras, whose interrelation is expected to be crucial for a better understanding of the complete structure of the Kac Moody algebra.

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Correspondence to: rwgebert@x4u2.desy.de or nicolai@x4u2.desy.de
1. Introduction

Affine Kac Moody algebras (see [29, 25] and references therein), which first appeared in physics in the guise of (two-dimensional) current algebras, have come to play an increasingly important role in string theory and conformal field theory as well as other branches of mathematical physics. By contrast, Kac Moody algebras based on indefinite Cartan matrices have not yet found applications in physics. In view of the scarcity of results about such algebras, it is remarkable that they have nevertheless been suggested as natural candidates for the still elusive fundamental symmetry of string theory (and hence of nature). Being vastly larger than affine Kac Moody algebras, Kac Moody algebras of indefinite type might certainly be "sufficiently big" for a unified and background independent formulation of string (field) theory, but an even more compelling argument supporting such speculations is the intimate link that exists between Kac Moody algebras and the vertex operator construction of string theory (this connection has been known for a long time [1, 28]). More specifically, it has been established that the generators making up a Kac Moody algebra of finite or affine type can be explicitly realized in terms of tachyon and photon emission vertex operators of a compactified open bosonic string [16, 24]. On the basis of these results, it has been conjectured that generalized Kac Moody algebras of indefinite type might not only furnish new symmetries of string theory, but might themselves be understood in terms of string vertex operators associated with the higher excited (massive) states of a compactified bosonic string [24, 14]. Despite its great appeal, however, this idea has not led to a truly satisfactory understanding of these Kac Moody algebras until now.

Disregarding possible physical applications in string theory, very little is known about indefinite Kac Moody algebras beyond their mere existence and the remarkable result that the Weyl-Kac character formula continues to hold for them [36, 29]. The basic problem here is the proliferation of timelike roots (having negative (length)) and the concomitant exponential growth in the dimension of the corresponding root spaces. For a limited number of cases, and in particular for roots of level two at most\(^1\), one knows explicit multiplicity formulas counting the dimension of the root spaces [30], but the complete root multiplicities are not known for a single Kac Moody algebra of indefinite type (root multiplicities can be determined in principle from the Peterson recursion formula [31], but this formula quickly becomes too unwieldy for practical use). Unfortunately, the available results have not shed much light on the structure of the corresponding root spaces, and, in contrast to affine Kac Moody algebras, a manageable representation of the root space elements has not been found so far. In an interesting recent development (more concerned with understanding the monster group than with applications in physics), complete and explicit multiplicity formulas were derived for the so-called fake monster Lie algebra based on the 26-dimensional Lorentzian even self-dual lattice $II_{25,1}$ [4]; this algebra is, however, not a conventional Kac Moody algebra in that it has imaginary simple roots beside the

\(^1\) The notion of "level of a root" is defined in Sect. 4.1.