1D-Quasiperiodic Operators. Latent Symmetries

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Abstract. A large class of discrete quasiperiodic operators is shown to be decomposed into orbits of $SL(2, \mathbb{Z})$ action with equal densities of states. Moreover under some natural assumptions all nontrivial representatives of the mentioned action transform operators with pure point spectrum into those with absolutely continuous spectrum. Some applications of these results are presented.

1. Introduction

Let us consider the general family of quasiperiodic operators acting in $l^2(\mathbb{Z})$ through the formula

$$(H \Psi)_l = \sum_{j=-\infty}^{\infty} \Psi_{l-j} f_{j}(\Phi + lx), \quad f_j \in C(S^1), \quad x \in \mathbb{R} \setminus \mathbb{Q}. \quad (1)$$

This paper is an attempt of outlook on this class as a whole in order to reveal which almost periodic features are responsible for various spectral properties and thus to provide some old results with a new understanding.

During the 80's there was a great amount of interest in almost periodic operators. The balance between generality and concreteness was periodically driven to either side. Perhaps the most attention was paid to investigations of the almost-Mathieu operator

$$(H \Psi)_l = \Psi_{l+1} + \Psi_{l-1} + \lambda \cos(\alpha l + \Phi) \Psi_l \quad (2)$$

and related models.

The almost-Mathieu equation arose from the model of Bloch electron in the uniform magnetic field [1] and has been studied extensively both from physical and mathematical points. It seemed to have very interesting properties but (in comparison with a related 2D-operator) a simple easy to handle form.

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The rigorous results concerning almost-Mathieu operator can be divided into two classes. Those which hold for more general operator families \([2-5]\) etc., and those which use essentially the almost-Mathieu specificity \([6-10]\). The theorems of the first class concern usually the nicer spectral properties but cover mainly the considerably smaller sets of parameters than the results of the second one. The specificity used is the symmetry of the almost-Mathieu family with respect to the Fourier transform (the "Aubry duality") which gives rise to proving the positivity or vanishing of Lyapunov exponents \(\gamma(E, \lambda)\) for corresponding values of \(\lambda\). This implies that the spectral measure does not contain an absolutely-continuous component for \(\lambda > 2\) and there are no pure point spectrum for \(\lambda < 2\). We will call such results as theorems "up to" the singular-continuous spectrum because clearing up whether there exists a singular-continuous spectrum for \(\lambda + 2\) and the proof of the believable pure singular-continuity of the spectral measure for \(\lambda = 2\) will give a complete description of the spectrum.

The duality argument at first appeared in \([6]\) as a quasitheorem and was proved rigorously in \([7]\) and \([8]\) by different methods. It has been uncertain what is the nature of the duality and whether this argument can be extended to the wider classes of almost-periodic problems.

In this paper we will show that symmetry properties of the whole class of 1D almost-periodic operators are caused by the existence of some gauge group for related 2D Hamiltonians. Thus a physically intuitive connection between almost-periodic and 2D operators in a uniform magnetic field becomes a mathematical fact which enables one to find a group of transforms (including the Fourier transform) that preserves the integrated density of states. These are the results of Theorems 1 and 2. Theorem 3 shows that under some natural assumptions every orbit of the above mentioned group action includes at most one isospectral class of operators with a pure point spectrum, all other operators in the orbit possessing pure absolutely-continuous spectrum. Theorem 4 is an example of application of Theorem 3. Theorem 5 relying on Theorem 2 describes the spectral properties up to the singular-continuous spectrum of some operator family.

Thus the essence of our method is that deriving 1D quasiperiodic operator from the 2D Hamiltonian one should not forget the initial problem. We remark that all known proofs of the Aubry duality have in fact the two-dimensional nature. All the latent symmetries become explicit when regarding the related 2D operator.

Let us give now the precise formulations.

We consider an operator \(H\) of the form (1). It is connected with the matrix of the Fourier coefficients \(B = B(k, j) = \|b_{k, j}\|\) appearing in the expansion of the functions

\[
f_j(x) = \sum_{j= -\infty}^{\infty} b_{k, j}\exp(ik(x - ja/2)).
\]

We will use the notation \(H_{b, \Phi}^{\#}\) for operator (1). (The indices \(x, \Phi\) would be ignored in all objects connected with \(H_{b}\) if it would not lead to ambiguity; in order to avoid rather cumbersome notations we will sometimes use \(b_l\) for \(b_{k, j}\) where \(l = (k, j) \in \mathbb{Z}^2\). Sometimes it will be convenient to deal with a diagram corresponding to the matrix \(B\), which can be built by omitting all zero coefficients \(b_{k, j}\) and providing all nonzero \(b_{k, j}\) with vectors going from the point \((0, 0)\) to the point \((k, j)\). Denote by \(\Sigma\) the set of operators \(H_{b}\) such that \(b_l = b_{l, -1}\) and \(b_l \in l^2(\mathbb{Z}^2)\). Let \(H_{b}^{\#}(A)\) be the operator \(H_{b}\) restricted to \(l^2(-A, A) \subset l^2(\mathbb{Z})\). Denote, as usual, by \((2A + 1)^N_{b}(\lambda; A)\) the number of eigenvalues \(\lambda(A)\) of \(H_{b}(A)\) which are less than \(\lambda\), i.e. \(\lambda(A) \leq \lambda\).