THE STABILITY OF THE COMPLETENESS
AND MINIMALITY IN L^2 OF A SYSTEM
OF EXPONENTIAL FUNCTIONS

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Let the sequences \( \{\lambda_n\} \) and \( \{\alpha_n\} \) of complex numbers satisfy the conditions: 1) \( \sup |\text{Im} \lambda_n| = h < \infty \); 2) the number of points \( \lambda_n \) in the rectangle \( |t - \text{Re} z| \leq 1, |\text{Im} z| \leq h \) is uniformly bounded with respect to \( t \in (-\infty, \infty) \); 3) \( \{\alpha_n\} \in l^p \) for some \( p < \infty \). Then the systems \( \{\exp(i\lambda_n x)\} \) and \( \{\exp(i(x\lambda_n + \alpha_n))\} \) are simultaneously complete or noncomplete (minimal or nonminimal) in \( L^2(-a, a) \) \( (a < \infty) \).

W. O. Alexander and R. Redheffer [1] and J. Elsner [2] have investigated the questions of the stability of the completeness and of the minimality of systems of exponential functions in the spaces \( L^p = L^p(-\sigma, \sigma), p \in [1, \infty], \sigma \in (0, \infty) \). Alexander and Redheffer have proved that if the sequences \( \{\lambda_n\} \) and \( \{\alpha_n\} \) of complex numbers satisfy the condition

\[
\sum \frac{|\alpha_n|}{(1 + |\text{Im} \lambda_n| + |\text{Im}(\lambda_n + \alpha_n)|)^{p-1} + q^{-1}} < \infty,
\]

then the systems \( \{\exp(i\lambda_n x)\} \) and \( \{\exp(i(x\lambda_n + \alpha_n))\} \) are simultaneously complete or noncomplete (minimal or nonminimal) in \( L^p \). When the points \( \lambda_n \) lie in some horizontal strip, that is, when \( \sup |\text{Im} \lambda_n| < \infty \), the condition (0) becomes the condition \( \{\alpha_n\} \in l^1 \), which is also given in [2].

In this note, by considering the space \( L^2 \) and by imposing on \( \{\lambda_n\} \) in addition to the condition \( \sup |\text{Im} \lambda_n| < \infty \), a regularity condition, we significantly extend the class of admissible "perturbations."

**THEOREM.** Let the sequences \( \{\lambda_n\} \) and \( \{\alpha_n\} \) of complex numbers be such that 1) \( \sup |\text{Im} \lambda_n| = h < \infty \), 2) the number of points \( \lambda_n \) in the rectangle \( |t - \text{Re} z| \leq 1, |\text{Im} z| \leq h \) is bounded by some number independent of \( t \in (-\infty, \infty) \), and 3) \( \{\alpha_n\} \in l^p \) for some \( p < \infty \). Then the systems \( \{\exp(i(x\lambda_n + \alpha_n))\} \) and \( \{\exp(i\lambda_n x)\} \) are simultaneously complete or noncomplete (minimal or nonminimal) in \( L^2 \).

For \( p = \infty \) the theorem is no longer true, no matter how small we take the norm \( \|\{\alpha_n\}\|_\infty \). This is shown by the examples of the system \( \{\exp(i(\lambda_n x))\}_{n=1}^\infty \), with 1) \( \lambda_n = n - (1/4) \text{sgn } n \), 2) \( \lambda_n = n - ((1/4) - \epsilon) \text{sgn } n \), \( 3) \lambda_n = n + (1/4) \text{sgn } n \), and \( 4) \lambda_n = n + ((1/4) + \epsilon) \text{sgn } n \). It follows from the results of N. Levinson [3] that the system 1) is not minimal in \( L^2(-\pi, \pi) \) but the system 2) is minimal in \( L^2(-\pi, \pi) \) for all sufficiently small \( \epsilon > 0 \); the system 3) is complete in \( L^2(-\pi, \pi) \) but the system 4) is noncomplete in \( L^2(-\pi, \pi) \) for all sufficiently small \( \epsilon > 0 \). This circumstance significantly distinguishes (in the sense of stability) the completeness (minimality) of the system \( \{\exp(i\lambda_n x)\} \) from the Riesz basis of the same form, because (see [4]) if the system \( \{\exp(i\lambda_n x)\} \) forms a Riesz basis in \( L^2 \), then there exists an \( L > 0 \) with the property: as soon as \( \sup |\alpha_n| < L \), the system \( \{\exp(i(x\lambda_n + \alpha_n))\} \) is also a Riesz basis in \( L^2 \).

We put \( \mu_n = \lambda_n + \alpha_n \). We assume that the sequence \( \{\lambda_n\} \) is enumerated in the order of nondecreasing real parts, and \( 0 \not\in \{\lambda_n\} \). 0 \not\in \{\mu_n\} \) (the latter does not reduce the generality). Let \( \text{Re } \lambda_n > 0 \) for \( n > 0 \).

We introduce the product

\[
F(z) = \lim_{R \to \infty} \prod_{n \leq R} \left(1 - \frac{z}{\mu_n}\right) \left(1 - \frac{z}{\lambda_n}\right)^{-1}.
\]


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It is easily seen that under the conditions of the theorem it is convergent everywhere apart from the set \{λ_n\}.

**Lemma.** We assume that the conditions of the theorem hold. If \(|H| > h + \sup |a_n|\), then \(|F(s)| < C(H) < \infty\) on the straight line \(Im z = H\).

Let \(G(z)\) denote the product (1) with \(\lambda_n\) (respectively, \(\mu_n\)) replaced by \(Re \lambda_n\) (respectively, \(Re \mu_n\)). It is clear from the proof of a lemma of the author ([5], pp. 1106–1109) that, for all \(\delta > 0\), the following inequality holds outside circles of radius \(\delta\) with centers at the points \(\lambda_n, \mu_n, Re \lambda_n, Re \mu_n\):

\[
0 < A(\delta) < \left| \frac{F(t)}{\partial (\delta)} \right| < B(\delta) < \infty.
\]

Hence we can assume that the sequences \{\(\lambda_n\)\} and \{\(\alpha_n\)\} are real.

We first assume that \(\inf |\lambda_n - \lambda_m| = \gamma > 0\). Because \(\lim a_n = 0\) we have \(|a_n| < \gamma/2\) for \(n > N_0\). It is obvious that if we replace a finite number of elements in the sequences \{\(\lambda_n\)\} and \{\(\mu_n\)\} by the same number of other numbers, then the relation being proved is unchanged. Hence we assume that \(|a_n| < \gamma/2\) for all \(n\). In addition, we can assume that \(\alpha_n = 0\) for \(n \leq 0\), and \(\mu_n > 0\) for \(n > 0\).

We use the notation: \(I_n = (\min(\lambda_n, \mu_n), \max(\lambda_n, \mu_n))\), \(m(t)\) [respectively, \(n(t)\)] denotes the number of points \(\mu_n\) (respectively, \(\lambda_n\)) in the interval \((0, t)\), and \(\varphi(t) = m(t) - n(t)\). The intervals \(I_n\) are disjoint by construction, and

\[
\ln |F(x + iH)| = \sum_{n=1}^{\infty} \left( \ln \left| 1 - \frac{x + iH}{\lambda_n} \right| - \ln \left| 1 - \frac{x + iH}{\mu_n} \right| \right) = \sum_{n=1}^{\infty} \ln \left| 1 - \frac{x + iH}{\lambda_n} \right| \varphi(t).
\]

The function \(\varphi(t)\) is bounded, \(\varphi(t) = 0\) outside \(I_n\), and on \(I_n\) it has the values \(\pm 1\). Hence, when we integrate by parts, the term outside the integral is zero, and we obtain

\[
\ln |F(x + iH)| = \sum_{n=1}^{\infty} \left\{ \int_{0}^{\infty} \left( 1 - \frac{t-x}{(t-z)^2 + H^2} \right) \varphi(t) dt \right\} \psi(t) dt = \sum_{n=1}^{\infty} \int_{I_n} \left( 1 - \frac{t-x}{(t-z)^2 + H^2} \right) \psi(t) dt.
\]

Hence

\[
\ln |F(x + iH)| \leq \sum_{\text{min}^{-1}(\lambda_n, \mu_n)} |\lambda_n - z| + \sum_{\text{max}^{-1}(\mu_n, \lambda_n)} |\mu_n - z| + H^2.
\]

where \(\lambda_n^* \in I_n\) (by the theorem about the mean).

As is usual we take \(p^{-1} + q^{-1} = 1\). Because \(q > 1\),

\[
\inf_{n \in \mathbb{N}} |\lambda_n - \lambda_m| > 0, \quad a_n \to 0,
\]

we have

\[
\{\text{min}^{-1}(\lambda_n, \mu_n)\} \subseteq \mathbb{N}^*;
\]

by Hölder's inequality the first series at (2) is convergent. It remains for us to show (again by using Hölder's inequality) that

\[
\sup_{x \in (-\infty, \infty)} \sum_{m=\text{min}(\lambda_n, \mu_n)} \left| \frac{m^* - x}{(\lambda_n^* - z)^2 + H^2} \right| < \infty.
\]

Let us note that the convergence of the series (3) for fixed \(x\) follows from the same arguments that led us to the result \{\text{min}^{-1}(\lambda_n, \mu_n)\} \subseteq \mathbb{N}^*.

For a fixed \(x \in (-\infty, \infty)\) let \(K_m\) denote the segment \(x-1 + 2m \leq t \leq x + 1 + 2m\), and let \(n(x)\) denote the smallest integer \(m\) for which the segments \(K_m\) contain the points \(\lambda_n^*\). By our assumption the number of points \(\lambda_n^*\) on \(K_m\) is bounded by some number \(s\) independent of \(m\). Hence

\[
\sup_{x} \sum_{m=1}^{\infty} \frac{|\lambda_n^* - z|^2}{(\lambda_n^* - z)^2 + H^2} \leq \sup_{x} \sum_{m=1}^{\infty} \sum_{n \in K_m} \frac{|\lambda_n^* - z|^2}{(\lambda_n^* - z)^2 + H^2} \leq \frac{2}{H^2} + s \sup_{x} \sum_{m=1}^{\infty} \left( \frac{2m + 1}{(2m - 1)^2 + H^2} \right)^2 \leq C(s, q, H) + 2s \sum_{m=1}^{\infty} \left( \frac{2m + 1}{(2m - 1)^2 + H^2} \right)^2 < \infty,
\]

because \(q > 1\).