Shift Operators for the Quantum Calogero–Sutherland Problems via Knizhnik–Zamolodchikov Equation

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Abstract: We give a natural interpretation of the shift operators for Calogero–Sutherland quantum problem via KZ equation using Matsuo–Cherednik mappings. The explicit formulas for the inversions of these mappings and versions of shift operators for KZ equations are also found. As an application we show that the shift operator can be described via a factorization problem for an appropriate quantum integral (discriminant) of the Calogero system.

Introduction

The Calogero system\textsuperscript{[1]} describes the motion of $N$ particles on the line interacting with the potential $U_g^c(x) = g u^c(x)$,

$$u^c(x) = \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2} ,$$

where $x_i$ are the coordinates of the points, $g$ is a coupling constant. The value of $g$ is not essential in the classical case but it is in the quantum case. For $N = 2$ the corresponding Schrödinger equation

$$(-\Delta + U_g^c(x))\psi = E\psi ,$$

$\Delta = \partial_1^2 + \cdots + \partial_N^2$, $\partial_i \equiv \partial/\partial x_i$, can be reduced to the one-dimensional one:

$$\left(-\frac{d^2}{dy^2} + \frac{g}{y^2}\right)\varphi = \lambda \varphi , \quad y = x_1 - x_2 .$$

If we introduce a new constant $k$, such that

$$k(k-1) = g ,$$

...
then the operator $L_k = -\frac{d^2}{dy^2} + \frac{k(k-1)}{y^2}$ can be factorized as

$$L_k = -\left(\frac{d}{dy} + \frac{k}{y}\right)\left(\frac{d}{dy} - \frac{k}{y}\right).$$

(0.4)

Let us remark that

$$L_{k+1} = -\left(\frac{d}{dy} - \frac{k}{y}\right)\left(\frac{d}{dy} + \frac{k}{y}\right),$$

(0.5)

which implies the following relation for the operator $\mathcal{D}_k = \frac{d}{dy} - \frac{k}{y}$:

$$L_{k+1} \mathcal{D}_k = \mathcal{D}_k L_k.$$  

(0.6)

This means that the operator $\mathcal{D}_k$ maps (formal) eigenfunctions of the operator $L_k$ to eigenfunctions of $L_{k+1}$: if $L_k \phi = \lambda \phi$ and $\tilde{\phi} = \mathcal{D}_k \phi$, then $L_{k+1} \tilde{\phi} = L_{k+1} \mathcal{D}_k \phi = \mathcal{D}_k L_k \phi = \lambda \tilde{\phi}$.

It turns out that such operators exist for all $N$. In the most general case this was proven by Opdam [6], who considered together with Heckman the trigonometric version of Calogero system (Sutherland system) and its generalization first proposed by Olshanetsky and Perelomov (see [2–6]).

The Sutherland system describes the motion of particles on the line (circle, if $\omega$ is purely imaginary) interacting with the potential $U_{\Delta, \omega} = g u_{\omega}^2$,

$$u_{\omega}^2(x) = \frac{\omega^2}{\sum_{i<j} \frac{1}{\sinh^2 (x_i - x_j) \omega}}.$$  

(0.7)

In the limit $\omega \to 0$ one has the Calogero potential (0.1), so $u_{0}^2 = u^2$.

Following Opdam, we will call such operators $\mathcal{D}_k$ shift operators. They are differential operators with highest term $\prod_{i<j} (\partial_i - \partial_j)$, which satisfy the relation

$$L_{k+1} \mathcal{D}_k = \mathcal{D}_k L_k.$$  

(0.8)

for $L_k = -\Delta + k(k-1) u_{\omega}^2$, $L_{k+1} = -\Delta + k(k+1) u_{\omega}^2$.

The explicit form of the shift operator $\mathcal{D}_1$ for any $N$ was first found in 1988 by Chalykh and one of the authors in [8], where the problem of supercomplete commutative rings of partial differential operators was discussed. It turned out that the shift operators $\mathcal{D}_k$ for integer $k$ play a very important role in this problem (see [8, 9]). Other very interesting applications of shift operators (e.g. the proof of some of Macdonald's conjectures) were found by Opdam in [7].

In 1990 Heckman used the so-called Dunkl operator to give the explicit form of the shift operators in the most general case [10, 11]. The Dunkl operator [12] looks very much like the differential operator of the Knizhnik–Zamolodchikov (KZ) equation, which appeared in conformal field theory in 1984 [13]1. This hints at a possible relation between KZ and Calogero–Sutherland (CS) equations. The explicit form of such relations was discovered by Matsuo [15]. The result of Matsuo was generalized and extended by Cherednik in [19].

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1 We would like to remark also the resemblance of all these operators with Moser's $L$-operator for Calogero–Sutherland system (see [14]).