CLASSES OF SUBMERSIONS OF RIEMANNIAN MANIFOLDS WITH COMPACT FIBERS‡

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In Riemannian geometry and its applications, the most popular is the class of Riemannian submersions (and foliations) [1–4] which are characterized by simplest mutual disposition of fibers. The purpose of the present article is to introduce other, more general, classes of submersions of Riemannian manifolds which, as well as the class of Riemannian submersions, are described by simple local properties of configuration tensors and to begin their study.

Given a submersion \( \pi : M \to \overline{M} \) of differentiable manifolds with compact connected fibers and any metric on \( M \), we define a metric on the base with the help of the \( L^2 \)-norm of horizontal fields. In this case \( T\overline{M} \) becomes a subbundle of some larger bundle \( \overline{M} \). The main class of totally geodesic submersions introduced in the article (Definition 1) corresponds to the metrics on \( M \) with simplest disposition of \( T\overline{M} \) in \( \overline{M} \). In the article we obtain a criterion for such submersions (Corollary 1); existence is proved by means of the product with a metric varying along fibers (Example 2). To study totally geodesic submersions, we use ideas from the theory of Riemannian submersions and submanifolds with degenerate second form (Theorems 1 and 2 and Corollary 4).

Foliations modeled by totally geodesic submersions (see equality (13)) are of interest too, but we leave them beyond the scope of the article.

1. Fundamental Equations of a Submersion of Riemannian Manifolds with Compact Fibers

1.1. We recall that a smooth mapping \( \pi : M \to \overline{M} \) between differentiable manifolds \( M \) and \( \overline{M} \) is said to be a submersion if the differential \( \pi_*(m) : T_m M \to T_{\pi(m)} \overline{M} \) is a surjection of tangent spaces at every point \( m \in M \). The manifold \( M \) is called the total space and \( \overline{M} \), the base. In the article we consider submersions with compact connected fibers \( \{L_b = \pi^{-1}(b)\}_{b \in \overline{M}} \); therefore, by Ehresmann's theorem [5], they are bundles; i.e., the following conditions are satisfied:

(1) \( \{L_b\} \) are diffeomorphic to a fixed manifold \( L \);

(2) for every point \( b \in \overline{M} \), there exist a neighborhood \( U_b \) and a diffeomorphism \( \varphi : \pi^{-1}(U_b) \to U_b \times L \), where \( b' \in U_b \).

Furthermore, in what follows \( M \) is endowed with a Riemannian metric and \( \dim \overline{M} > 1 \). The vectors and fields on \( M \) that are tangent and perpendicular to fibers are referred to as vertical and horizontal, respectively. A horizontal field \( Y = \pi_*^{-1}(\overline{Y}) \) along a fiber \( L_b \), where \( \overline{Y} \in T_b \overline{M} \), is referred to as a basis field. We agree to denote by \( x, x_1, \) and \( x_2 \) vertical vectors; by \( Y, Z, U, V \), and \( W \), horizontal vectors and basis fields; by \( s, s_1, \) and \( s_2 \), horizontal fields; by \( T \), the orthoprojection of \( TM \) onto \( TL \); and by \( \perp \), the complementary orthoprojection.

The horizontal metric connection \( \nabla \) and the horizontal curvature operator \( \tilde{R} \) of a submersion are given by the formulas

\[
\tilde{\nabla}_YZ = (\nabla_Y Z) \perp, \quad \tilde{R}(Y, Z)U = \tilde{\nabla}_{[Y, Z]}U - [\tilde{\nabla}_Y, \tilde{\nabla}_Z]U;
\]

moreover, for \( \tilde{R} \) all symmetries of a curvature tensor are valid as well as both the Bianchi identities [2].

‡) This work was supported by the Russian Foundation for Fundamental Research (Grant 94–01–00271).
The configuration tensors $A$ and $T$ of a submersion $\pi$ are given by the formulas [2]

$$
A_Y Z = (\nabla_Y Z)^T, \quad A_Y x = (\nabla_Y x)^\perp, \quad T_{x_1} x_2 = (\nabla_{x_1} x_2)^\perp, \quad T_x Y = (\nabla_x Y)^T.
$$

The tensor $T$ presents the second fundamental form of fibers, so the equality $T = 0$ means that they are totally geodesic. Since $[x, Y]^\perp = 0$ [2], the basis fields are characterized by the equation

$$
(\nabla_x Y)^\perp = A_Y x \quad (x \in TL).
$$

Lemma 1 [2]. The formulas

$$
A_Y Z - A_Z Y = [Y, Z]^T, \quad (A_Y x, Z) = -(A_Y Z, x),
$$

are valid along with the following formulas known as the Gauss equation, Codazzi equation, and Ricci equation:


\[ (R(Y, Z)W, x) = ((\nabla_Z A)Y W - (\nabla_Y A)Z W, x) + (A_Y Z - A_Z Y, T_x W), \]

\[ (R(Y, x_1)Z, x_2) = ((\nabla_{x_1} A)Y Z + A_{A_Y x_1} Z, x_2) + ((\nabla_Y T)_{x_1} x_2 + T_{x_1} Y x_2, Z). \]

Below we will also use the second fundamental form $A_Y^+ Z = (A_Y Z + A_Z Y)/2$ and the integrability tensor $A_Y^- Z = (A_Y Z - A_Z Y)/2$ of the distribution $TL^\perp$. They have the following geometric meaning: $A^+$ is the second fundamental form, at a point $m$, of the submanifold composed of the geodesics in $M$ tangent to the subspace $T_m L^\perp$; $A^- = 0$ with the distribution $TL^\perp$ integrable (see Lemma 1). For instance, alternating (2b) with respect to $Y$, $Z$, and $W$ and involving the first Bianchi identity, we obtain the following useful identity (cf. Subsection 2.1 below):

$$
(\nabla_Y A^-)Z W + (\nabla_Z A^-)W Y + (\nabla_W A^-)Y Z = 0.
$$

Every Riemannian submersion specified by the equality $A^+ = 0$ has fibers equidistant to each other; i.e., basis fields have constant length; moreover, its base is endowed with a metric $(,)^*$ such that $\pi_*(m) : T_m L^\perp \to T_{\pi(m)} \tilde{M}$ are isometries (see a generalization in [6]). Hopf bundles serve as examples. But even for odd-dimensional spheres other classes of submersions are of interest, for instance the skew Hopf bundles [7] which are useful in the study of Blaschke manifolds.

1.2. Along a fixed fiber $L_b$, the horizontal vector fields form a vector space $\tilde{M}_b$ with the scalar product

$$
\langle s_1, s_2 \rangle_b = \int_{L_b} (s_1, s_2) d\text{vol}.
$$

The basis fields along $L_b$ form an $(\dim \tilde{M})$-dimensional Euclidean subspace $T_b \tilde{M} \subset \tilde{M}_b$; therefore, $\tilde{M}$ with metric $(,)$ is a Riemannian manifold. In the particular case when $\tilde{M}$ is the manifold of geodesics of a $C^1$-manifold, the metric $(,)$ on the base was considered in [8]. In contradistinction to O'Neill's method [1] which is applicable only to submersions with equidistant disposition of fibers, the metric $(,)$ is defined on $\tilde{M}$ for an arbitrary metric on $M$. In the case when the fibers are equidistant, the metric $(,)$ is conformally equivalent to O'Neill's metric $(,)^*$: over a point $m \in \tilde{M}$, it appears after multiplication by the volume of the fiber over the point.

Since $\pi : M \to \tilde{M}$ is a smooth bundle, the spaces $\{\tilde{M}_b\}$ are fibers of the vector bundle $p : \tilde{M} \to \tilde{M}$; moreover, $\tilde{M} = \cup \{\tilde{M}_b\}$ and $T\tilde{M}$ (the vectors are basis fields) is a subbundle of $\tilde{M}$. We introduce