ON DIFFERENTIABILITY OF SOLUTIONS TO HOMOGENEOUS ELLIPTIC EQUATIONS OF DIVERGENCE TYPE

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Introduction. In the article we consider the functions of class $W^{1}_{2,\text{loc}}(U)$ (where $U$ is a domain in $\mathbb{R}^{n}$) which are generalized solutions to the elliptic equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_{i}} \left( \sum_{j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_{j}} \right) = 0. \tag{1}$$

As regards the coefficients $a_{ij}(x)$, we assume that they are measurable and, for almost all $x \in U$ and all $\xi \in \mathbb{R}^{n}$, satisfy the inequalities

$$\lambda_{1} |\xi|^{2} \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_{i} \xi_{j} \leq \lambda_{2} |\xi|^{2}, \tag{2}$$

where $0 < \lambda_{1} \leq \lambda_{2} < \infty$.

Assigning

$$\Delta(x) = \sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(x_{0})|,$$

we assume

$$\sup_{|x-x_{0}| \leq r} \Delta(x) = \omega(r), \tag{3}$$

where $\omega(r) \to 0$ as $r \to 0$. Further, we suppose that there exist $k > 0$ and $\alpha > 0$ such that

$$\int_{0}^{k} \frac{\omega(t)}{t^{1+\alpha}} dt < \infty. \tag{4}$$

In the article we prove the following theorems:

**Theorem 1.** Under the above-made assumptions, the function $u(x)$ is differentiable at the point $x_{0}$ in the sense of $W^{1}_{2}$.

**Corollary 1.** The function $u(x)$ is differentiable at $x_{0}$ in the conventional sense.

**Theorem 2.** Suppose that $|u'(x)| \leq M$ and $\Delta(x) \leq M$ almost everywhere in $U$ and there exists an $\alpha > 0$ such that

$$\int_{U} \frac{\Delta(x)}{|x-x_{0}|^{n-\alpha}} dx < \infty.$$

Then the function $u(x)$ is differentiable at $x_{0}$. 

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Corollary 2. Suppose that \( f \) is a quasi-isometric mapping and there exists an \( \alpha > 0 \) such that
\[
\int_U \frac{L_f(x) - 1}{|x - x_0|^{n+\alpha}} \, dx < \infty,
\]
where \( L_f(x) \) is the quasi-isometry coefficient of \( f \) at \( x \). Then \( f \) is differentiable at \( x_0 \).

1. Preliminaries. We let \( U \) be a domain in \( \mathbb{R}^n \), let \( B(r) \) denote the ball with center the origin and radius \( r \), and let \( S(r) \) denote the boundary sphere of \( B(r) \). Given a function \( f : U \to \mathbb{R} \), we assign
\[
\nabla f(x) = f'(x) = \left( \frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right),
\]
\[
\|f\|_{L^2(U)} = \left( \int_U |f(x)|^2 \, dx \right)^{1/2}, \quad \|f\|_{W^2_2(U)} = \|f\|_{L^2(U)} + \|f'\|_{L^2(U)}.
\]

We say that a function \( f : U \to \mathbb{R} \) is differentiable in the sense of convergence in \( W^2_2 \) at a point \( a \in U \) if there exists a linear mapping \( L : \mathbb{R}^n \to \mathbb{R} \) such that, for all sufficiently small \( h, 0 < h \leq h_0 \), the function \( r_h : x \in B(1) \mapsto (1/h)[f(a + hx) - f(a) - hL(x)] \) is defined on the ball \( B(1) \), and belongs to the space \( W^2_2 \), and \( \|r_h\|_{W^2_2(B(1))} \to 0 \) as \( h \to 0 \). The linear mapping \( L \) is called the \( W^2_2 \)-differential of \( f \) at \( a \). The function is differentiable in the conventional sense in case \( \|r_h\|_{C[\overline{B(1)}]} \to 0 \) as \( h \to 0 \).

A function \( u : U \to \mathbb{R} \) is said to be a generalized solution to equation (1) if the equality
\[
\int_U \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dx = 0
\]
holds for every function \( \varphi \) of class \( W^2_2 \) with compact support in \( U \).

As is known (see, for instance, [1]), a generalized solution to equation (1) exists, is continuous, and satisfies the Hölder condition in each strictly interior subdomain; in particular, if \( u(x) \) is a generalized solution to equation (1) in \( B(R) \) and \( u(0) = 0 \), then there exists a \( \gamma > 0 \) such that \( |u(x)| \leq cr^\gamma \) for \( |x| = r < R \).

The following theorem ensues from [2].

**Theorem 3.** Suppose that \( u(x) \) is a solution to equation (1) and that \( u(x) \) is differentiable at \( x_0 \) in the sense of \( W^2_2 \). Then \( u(x) \) is differentiable at \( x_0 \) in the conventional sense.

We shall need the following result of [3]:

**Theorem 4.** Suppose that \( a_{ij}(x) = \delta_{ij} + \epsilon_{ij}(x) \), where \( \epsilon_{ij}(x) = \epsilon_{ji}(x) \) are measurable functions such that
\[
\left| \sum_{i,j=1}^n \epsilon_{ij}(x) \xi_i \xi_j \right| \leq \epsilon |\xi|^2, \quad \epsilon < 1,
\]
for almost all \( x \in U \) and all \( \xi \in \mathbb{R}^n \). Then there exists a \( k = k(n) > 0 \) such that each solution \( u \) to equation (1) belongs to \( W^1_{p,\text{loc}}(U) \) for every \( p \in (1, 2 + k \log(1/\epsilon)) \).

The proof of the assertion relates to the reverse Hölder inequality. It is shown in [4] that the estimate \( p < c/\epsilon \) for the integrability exponent \( p \) is obtainable for \( \epsilon \) sufficiently small.

We now present some estimates for harmonic functions. For every harmonic function \( h(x) \) in the ball \( B(x_0, r) \), the following inequality holds:
\[
|\nabla h(x_0)| \leq cr^{-n/2} \left( \int_{B(x_0, r)} |\nabla h(y)|^2 \, dy \right)^{1/2}.
\]

(5)