ON NORMAL SOLVABILITY OF THE OPERATOR OF EXTERIOR DERIVATION ON WARPED PRODUCTS†

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Given a Riemannian manifold $M$, a number $p \geq 1$, and a positive continuous function $\sigma$ on $M$, the de Rham $L^p$-complex $(L^p_p(M, \sigma), d)_{k \in \mathbb{Z}}$ is defined comprising those differential forms on $M$ whose moduli are integrable in power $p$ with weight $\sigma^p$. The differentials $d$ of this complex are the exterior derivatives. These operators are densely defined and closed. The cohomology spaces of the de Rham complex are topological vector spaces, possibly nonseparated. These spaces are called the $L^p$-cohomologies of the manifold $M$ with weight $\sigma$ and denoted by $H^k_p(M, \sigma)$, while the spaces $\overline{H}^k_p(M, \sigma) = H^k_p(M, \sigma)/\{0\}$ are referred to as the reduced $L^p$-cohomologies of $M$.

If $M$ is a compact manifold then, for every $p \geq 1$ and an arbitrary weight $\sigma$, the $L^p$-cohomologies $H^k_p(M, \sigma)$ coincide with the conventional real (singular) cohomologies of the manifold and hence are finite-dimensional. Thereby for every $k$ the exterior derivative $d : L^k_p(M, \sigma) \to L^{k+1}_p(M, \sigma)$ on a compact manifold $M$ is normally solvable, i.e., it has closed range.

In the case of $p = 2$ and $\sigma \equiv 1$, J. Cheeger [1] established normal solvability of the operator $d$ on the Riemannian manifolds representing regular parts of compact pseudomanifolds, in particular, on a cone over a compact manifold $Y$. In [2], Cheeger’s result relevant to a cone was generalized to the case of warped cylinders $M = [a, b) \times_f Y$ with a monotone warping function $f$ and a bounded half-interval $[a, b)$; moreover, this was performed for all $p \in (1, \infty)$. In [3], necessary and sufficient conditions for normal solvability of the operator $d : L^k_p(M, \sigma) \to L^{k+1}_p(M, \sigma)$ on the manifold $M = [a, b) \times_f Y$ were found in the case when $-\infty < a < b \leq \infty$, $H^\infty_p(Y) = \overline{H}^\infty_p(Y)$, $\lim_{x \to b} f(x) = \infty$, and the weight function $\sigma$ on $M$ depends only on the variable $x \in [a, b)$. Using the Künneth formula for $L^p$-cohomologies [4, 5], some criterion for normal solvability of the operator $d$ can be derived for an arbitrary warped product $X \times_f Y$ with a bounded function $f$ and a compact $Y$.

In [6], methods were developed for proving nontriviality of $L^p$-cohomologies of warped products. In the present article, we put forward a new view on these methods which allows us to improve the methods and, on their grounds, derive a series of necessary conditions for normal solvability of the operator $d$ on a warped product. Given a Riemannian manifold $X$ and two weight functions $\tau$ and $\xi$ on $X$, we define the cochain complex $(R^i_p(X, \tau, \xi), \delta)$, where

\[ R^i_p(X, \tau, \xi) = L^{i-1}_p(X, \tau) \times L^i_p(X, \xi), \quad \delta(\omega_1, \omega_2) = (\omega_2 - d\omega_1, d\omega_2). \]

The complex $R_p(X, \sigma_{j+1}, \sigma_j)$ with the weights $\sigma_{j+1} = \sigma \cdot f^{n/p-j-1}$ and $\sigma_j = \sigma \cdot f^{n/p-j}$ turns out to be equivalent (with the shift of dimension by $j$) to some direct summand of the complex $L_p(X \times_f Y, \sigma)$. Therefore, the normal solvability of the operator $d : L^k_p(X \times_f Y, \sigma) \to L^{k+1}_p(X \times_f Y, \sigma)$ implies the normal solvability of the operators $\delta : R^{k-j}_p(X, \sigma_{j+1}, \sigma_j) \to R^{k-j+1}_p(X, \sigma_{j+1}, \sigma_j)$. In the article, we find some conditions for normal solvability of the operator $\delta : R^i_p(X, \tau, \xi) \to R^{i+1}_p(X, \tau, \xi)$; construct an exact sequence that binds the cohomologies of the complex $R_p(X, \tau, \xi)$ to those of the de Rham complexes $L_p(X, \tau)$ and $L_p(X, \xi)$; and compute the cohomologies and reduced cohomologies of the complex $R_p(X, \tau, \xi)$ for $X = [a, b)$. Also, we reveal a connection between the question of normal solvability of the exterior derivative $d$ on a warped cylinder $[a, b) \times_f Y$ and the embedding theorems for weighted Sobolev spaces on the half-interval $[a, b)$.

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§ 1. Banach Complexes $R_{p,1}(X, τ, ξ)$

A sequence $A = (A^i, d^i_A)_{i ∈ ℤ}$ of Banach spaces $A^i$ and closed linear operators $d^i_A : A^i → A^{i+1}$ is said to be a Banach complex if each operator $d^i_A$ is defined on a dense set $\text{dom} d^i_A$ in $A^i$ and $\text{Im} d^i_A ⊂ \text{Ker} d^{i+1}_A$. We shall use the conventional notations for cocycles, coboundaries, and cohomologies: $Z^iA = \text{Ker} d^i_A$, $B^iA = \text{Im} d^{i-1}_A$, $H^iA = Z^iA/B^iA$, and $H̅^iA = Z^iA/B̅^iA$. The norm of the Banach space $A^i$ induces a seminorm in $H^iA$ and a norm in $H̅^iA$. A cochain mapping $α$ from a Banach complex $A$ into a Banach complex $B$ is thought of as an arbitrary sequence $α^i : A^i → B^{i+j}$ (with $j$ fixed) of continuous linear mappings satisfying the following conditions: $α^i(\text{dom} d^i_A) ⊂ \text{dom} d^j_B$ and $α^{i+1} ∘ d^i_A = d^{i+j} ∘ α^i$ over $\text{dom} d^i_A$ for every $i ∈ ℤ$. A cochain mapping $α : A → B$ between Banach complexes induces continuous linear mappings $α^* : H^iA → H^{i+j}B$ and $α̅^* : H̅^iA → H̅^{i+j}B$.

We shall omit indices in the notations $d^i_A$ and $α^i$ wherever this creates no confusion.

Suppose that $η : A × B → ℝ$ is a continuous bilinear pairing between Banach spaces $A$ and $B$. If the correspondence $a → η(a, -)$ is a topological isomorphism from the Banach space $A$ onto the dual space $B^*$ of $B$, then we shall say that the space $A$ is dual to $B$ under the pairing $η$. Analogously, if the correspondence $b → η(-, b)$ is a topological isomorphism between the Banach spaces $B$ and $A^*$, then the space $B$ is said to be dual to $A$ under the pairing $η$. A family $η = (η^i : A^i × B^{N-i} → ℝ)_{i ∈ ℤ}$ (with $N$ fixed) of continuous bilinear pairings is said to be a pairing between Banach complexes $A$ and $B$ if $η^{i+1}(da, b) = (-1)^{i+1}η^i(a, db)$ for all $a, b ∈ \text{dom} d$. We say that the complex $B$ is dual to the complex $A$ under $η$ if, for every $i ∈ ℤ$ the space $B^{N-i}$ is dual to $A^i$ under $η^i$ and the operator $(-1)^{i+1}d^{N-i-1}_B$ is dual to $d^i_A$. By analogy, we say that the complex $A$ is dual to the complex $B$ under $η$ if, for every $i ∈ ℤ$, the space $A^i$ is dual to $B^{N-i}$ under $η^i$ and the operator $(-1)^{i+1}d^{N-i-1}_A$ is dual to $d^i_B$.

The following lemma is well known in the case of continuous linear operators (Serre’s lemma [7]).

**Lemma 1.** An arbitrary pairing $η$ between Banach complexes $A_1$ and $A_2$ induces continuous bilinear pairings $η^i : H^iA_1 × H^{N-i}A_2 → ℝ$. If one of the complexes $A_j$ is dual to the other under $η$ and all the spaces $A^i_j$ are reflexive, then the Banach spaces $H^iA_1$ and $H^{N-i}A_2$ are dual to each other under $η^i$.

**Proof.** If $η = (η^i)_{i ∈ ℤ}$ is a pairing between Banach complexes $A_1$ and $A_2$, then $η^i(a, b) = 0$ in each of the following two cases: (1) $a ∈ B^iA_1$ and $b ∈ Z^{N-i}A_2$; (2) $a ∈ Z^iA_1$ and $b ∈ B^{N-i}A_2$. Therefore, the pairings $η^i$ induce the pairings $η^i : H^iA_1 × H^{N-i}A_2 → ℝ$ that are obviously bilinear and continuous.

If the spaces $A^i_j$ are reflexive and one of the operators $d^i_{A_1}$ and $(-1)^{i+1}d^{N-i+1}_{A_2}$ is dual to the other, then the two are mutually dual [8, Theorem III.5.29]. In this case

\[
Z^iA_1 = \text{Ker} d^i_{A_1} = (\text{Im} d^{N-i-1}_{A_2})^⊥ = (B^{N-i}A_2)^⊥,
\]

\[
(Z^iA_1)^⊥ = (B^{N-i}A_2)^⊥⊥ = \overline{B^{N-i}A_2}.
\]

Analogously, $Z^{N-i}A_2 = (B^iA_1)^⊥$. Consequently,

\[
(H^iA_1)^* = (Z^iA_1/B^iA_1)^* = (B^iA_1)^⊥/(Z^iA_1)^⊥ = Z^{N-i}A_2/\overline{B^{N-i}A_2} = H^{N-i}A_2.
\]

The lemma is proven.

A closed densely defined operator $T$ is normally solvable if and only if the dual operator $T^*$ is normally solvable [8, Theorem IV.5.13]. For that reason, if one of the Banach complexes $A$ and $B$ is dual to the other then the operator $d^i_A$ is normally solvable if and only if the operator $d^{N-i-1}_B$ is normally solvable. In other words, for dual Banach complexes $A$ and $B$, the assertions that $H^{i+1}A = H^{i+1}A$ and $H^{N-i}B = H^{N-i}B$ are equivalent.