THE VARIATIONAL MAXIMUM PRINCIPLE
AND SECOND-ORDER OPTIMALITY CONDITIONS
FOR IMPULSE PROCESSES AND SINGULAR PROCESSES
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Introduction

In the present article we study a typical class of optimal control problems that admit of impulse-
trajectory extension \([1-3]\), i.e. the introduction of processes with discontinuous trajectories and con-
trols of impulse type. For the extended problem we obtain necessary and sufficient conditions for
a minimum on some set of sequences \([4]\) with the trajectory components converging in \(L_1\) and the
control components converging in the distribution sense. Analogous type of minimum and optimality
conditions make sense as regards the initial problem whose extremals present a version of the singular
mode. It is remarkable that the first order optimality criterion, i.e. the variational maximum principle
(VMP), if applied to singular processes, implies the classical principle rather than reduces to it.

We consider the following problem \((J, \mathcal{A})\) as initial:

\[
J(x, u) = l_0(b) \rightarrow \inf, \\
l_i(b) \leq 0, \quad i = 1, d(l), \quad k(b) = 0, \quad \text{ (1)} \\
\varphi_j(x(t)) \leq 0, \quad j = 1, d(\varphi), \quad \text{ (2)} \\
x = f(x) + G(x)u. \quad \text{ (3)}
\]

Here \(b = (x_0, x_1) = (x(t_0), x(t_1))\), the interval \(T = [t_0, t_1]\) is fixed, \(x\) is an absolutely continuous
function, \(u\) is a bounded measurable function, and the dimensions of \(x\), \(u\), and \(\kappa\) are equal to \(d(x), d(u), \text{ and } d(\kappa)\) respectively \((d(x) \text{ stands for the dimension of a vector } x)\). We denote by \(\mathcal{A}\) the set of
pairs of functions \((x, u)\) that satisfy conditions (1)-(3).

**Assumptions:**

1. The functions \(l_i, \kappa, \varphi_j, \text{ and } f\) are twice continuously differentiable while the
smoothness of the matrix \(G\) is greater by unity.

2. Phase conditions (2) are regular in the following sense: if the inequalities \(\varphi_j(x) \leq 0, \quad j = 1, d(\varphi), \)
are valid at a point \(x\) and the set \(I_{\varphi}(x) = \{j \mid \varphi_j(x) = 0\}\) is nonempty then there exists a vector
\(w \in R^{d(u)}\) such that \((\varphi_j(x), G(x)w) < 0 \forall j \in I_{\varphi}(x)\) (we denote by \(\langle \cdot, \cdot \rangle\) the inner product). The
assumption is required for justifying the VMP, whereas the passage to the second-order conditions is
justified under its strengthening as follows: the vectors \(G^*(x)\varphi_j(x), j \in I_{\varphi}(x), \) are linearly indepen-
dent if \(I_{\varphi}(x) \neq \emptyset\) (the star indicates the taking of transposition).

3. The many-dimensional system of partial differential equations

\[
\xi_w = G(\xi), \quad \xi|_{w=0} = y \quad \text{ (4)}
\]

(dim \(w = d(u)\)) has a global solution \(\xi(y, w)\) for every \(y \in R^{d(x)}\). In particular, if \(G_1, \ldots, G_{d(u)}\)
are the columns of the matrix \(G\) then the Frobenius complete integrability condition is satisfied:

\[G_{i\xi}(x)G_j(x) = G_{j\xi}(x)G_i(x) \forall x \in R^{d(x)}, \quad i, j = 1, d(u).\]

Since the problem \((J, \mathcal{A})\) is linear with respect to the unbounded control, the Pontryagin maximum
principle is insufficiently informative and we often find ourselves in a situation in which there is no
minimals in \(\mathcal{A}\). By this reason, we will consider the extension of the problem \((J, \mathcal{A})\) to the class of
impulse processes.

The Frobenius condition is typical of most methods for impulse-trajectory extension. Somewhat more general situations reduce to it with the help of a number of tricks [1-3]. Together with the growth condition \(|G(x)| \leq c(1 + |x|)\), the Frobenius condition guarantees validity of Assumption 3. The requirement of global solvability of system (4) can be withdrawn by conducting the further consideration in the domain of existence of an inextensible solution to system (4). For simplicity, we shall not consider such a restriction. We also point out that autonomy of the problem is not crucial for us.

In the present article we study, first, correctness of the constructive methods of extension [1-3] which are based on special transformations of the initial problem and provide a characterization for impulse processes which does not relate to the transformation. Second, we establish necessary and sufficient optimality conditions for an impulse process which correspond to the deciphering of the Pontryagin minimum conditions [4] for the transformed problem in terms of the initial problem. By now these questions remained open.

\section{The Extended Problem.}

\textbf{Statement of the Question of Deciphering}

For constructive description of the extended problem, we use the transformation that was proposed in [5]:

\[ \theta : (x, u) \rightarrow (y, w, h) \in AC^{d(x)} \times W_{1,\infty}^{d(u)} \times R^{d(w)}, \]

\[ w = u, \quad w(t_0) = 0, \quad h = w(t_1), \]

\[ y(t) = \xi(x(t), -w(t)), \quad t \in T. \]  

(5)  

Then the triple \( \nu = (y, w, h) \in \theta(\mathcal{A}) \) becomes a feasible process for the following problem (I, \mathcal{N}):

\[ I(\nu) = l_0(y_0, \xi(y_1, h)) \rightarrow \inf, \]

\[ l_i(y_0, \xi(y_1, h)) \leq 0, \quad i = 1, \overline{d(l)}, \quad \kappa(y_0, \xi(y_1, h)) = 0, \]

\[ \varphi_j(y_0) \leq 0, \quad \varphi_j(\xi(y_1, h)) \leq 0, \quad j = 1, \overline{d(\varphi)}, \]

\[ \varphi_j(\xi(y, w)) \leq 0, \quad j = 1, \overline{d(\varphi)}, \]

\[ \dot{y} = g(y, w). \]  

(7)

Here \((y_0, y_1) = (y(t_0), y(t_1)), g(y, w) = \xi_y^{-1}(y, w)f(\xi(y, w)), w \) is a control of class \( L_{\infty}, h \) is a parameter (i.e., we abandon connections (5)), and \( \mathcal{N} \) is the set of feasible triples \( \nu \) admitted by the conditions of the transformed problem.

The inverse transformation is determined on the image \( \theta(\mathcal{A}) \) by conditions (5) and the equality

\[ x(t) = \xi(y(t), w(t)), \quad t \in T. \]

(8)

We call a triple \( e = (x, w, h) \) a feasible impulse process if there is a process \( \nu = (y, w, h) \in \mathcal{N} \) such that equality (8) holds almost everywhere (a.e.) on \( T \); in this case we put \( x(t_0) = y(t_0) \) and \( x(t_1) = \xi(y(t_1), h) \) by definition. We denote the set of feasible processes by \( \mathcal{E} \) and denote the corresponding optimization problem by \( (J, \mathcal{E}) \).

It is obvious, in view of (6) and (8), that there is a bijection \( \theta : \mathcal{E} \rightarrow \mathcal{N} \) implying that the problems \( (J, \mathcal{E}) \) and \( (I, \mathcal{N}) \) are equivalent, \( J(e) = I(\theta(e)) \forall e \in \mathcal{E}, \) and (since \( \theta(\mathcal{A}) \subset \mathcal{N} \))

\[ \inf J(\mathcal{A}) \geq \inf J(\mathcal{E}) = \inf I(\mathcal{N}). \]

Moreover, strict inequality can take place even when local conditions (2) are absent in the initial problem [1, 2].