Explicit solutions are constructed for superized Toda lattices associated with systems of simple roots of classical Lie superalgebras. The completeness of flows and the asymptotic behavior of systems are studied.

0. In the present paper we consider one-dimensional nonclosed Toda lattices, constructed from systems of simple roots of simple Lie superalgebras $\mathfrak{sl}(m|n), \mathfrak{osp}(m|2n), \mathfrak{g}(n)$. Cf. [1, 5] for the information needed. We shall follow the general "Adler–Kostant scheme" on a Lie superalgebra described in [2]. Solutions for the Toda lattices associated with the superalgebras $\mathfrak{sl}(m|n), \mathfrak{osp}(m|2n)$ are expressed in terms of the "superminor" of the exponential of a matrix, depending on the initial data. For the classical nonclosed Toda lattice associated with the Lie algebra $\mathfrak{sl}(n)$, the solutions are described in analogous terms in [7]. We also investigate the completeness of the flows and the behavior of the system as $t \to \infty$.

The integrability of the Toda lattices associated with $\mathfrak{sl}(n|m)$ and $\mathfrak{osp}(m|2n)$ follows from the strong general theorem of Shander [4]. The case of the lattices associated with $\mathfrak{g}(n)$ is not handled by this theorem. In the present paper the integrability of this system is proved: integrals in involution are produced and explicit solutions are described.

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1. Let $\Lambda$ be a Grassman algebra with a finite number of generators, $\mathbb{E}(\Lambda) = (\mathbb{E} \otimes \Lambda)^\mathbb{C}$ be the Grassman hull of the linear superspace $\mathbb{E}$, $\mathbb{G}(\Lambda)$ be the Lie group with which the functor represented by the supergroup $\mathcal{G}$ associates the commutative superalgebra $\Lambda$; for example,

\[
\mathfrak{sl}(m|n)(\Lambda) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}(m,n;\Lambda)^\mathbb{C} ; \text{Ber} \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = 1 \right\},
\]

\[
\mathfrak{osp}(m|2n)(\Lambda) = \left\{ X \in \mathfrak{sl}(m|2n)(\Lambda) ; \alpha^t \mathcal{B} X \mathcal{B} X = \mathcal{B}^{-1} \right\},
\]

where

\[
\mathcal{B}_{m,n} = \begin{pmatrix} \varepsilon_m & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & -1_0 & 0 \end{pmatrix}, \quad \varepsilon_m = \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 0 \end{pmatrix}, \quad \varepsilon_m \in \text{Mat}(m).
\]
Let \( \mathfrak{g} \) be the Lie superalgebra of the Lie supergroup \( G \).

We shall identify \( \mathfrak{g} \) and \( \mathfrak{g}^* \) with the help of the Killing form \( \langle X, Y \rangle = \text{tr} X Y \) for \( \mathfrak{sl}(m|n) \) and \( \text{osp}(m|l|n) \), and for \( \mathfrak{g}(m) \) and \( \pi(\mathfrak{g}(m))^* \) with the help of the odd form

\[
\langle X, Y \rangle = \text{tr} X Y, \quad \text{where} \quad \text{tr}(A B) = \text{tr} B.
\]

(Here \( \pi \) is the change of parity functor.)

Each system \( \Pi \) of simple roots of the Lie superalgebra \( \mathfrak{g} \) defines on \( \mathfrak{g} \) a \( \mathbb{Z} \) -grading and thus a decomposition of \( \mathfrak{g} \) as a linear superspace on two algebras \( \mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_- \), where \( \mathfrak{g}_+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \), \( \mathfrak{g}_- = \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha \). In the group \( G(\Lambda) \) to these superalgebras there correspond connected subgroups \( \mathbb{N}_+(\Lambda) \) and \( \mathbb{Q}_-(\Lambda) \). Corresponding to this decomposition on \( \mathfrak{g} \) one introduces a new structure of Lie superalgebra with bracket

\[
[a_1 + b_1, a_2 + b_2] = [a_1, a_2] - [b_1, b_2], \quad a_1, a_2 \in \mathfrak{g}_+, \quad b_1, b_2 \in \mathfrak{g}_-.
\]

The Poisson bracket on the space \( \mathfrak{g}_0(\Lambda)^* \) is defined by the standard formula \( \{\varphi, \psi\}(\xi) = \langle \varphi, [\psi, \xi] \rangle \). This bracket is nondegenerate on orbits of the coadjoint representation of the group \( G_0(\Lambda) = \mathbb{N}_+(\Lambda) \times \mathbb{Q}_-(\Lambda) \).

By a superized generalized Toda lattice we shall mean an (ordinary) Hamiltonian system which for each Grassman algebra \( \Lambda \) is defined by the Hamiltonian \( h = \frac{1}{2} \langle \xi, \Lambda \rangle \) on the orbit \( f + O_{e} \) of the element \( f + e \), where \( f = \sum c_\alpha e_\alpha \in \mathfrak{g}_0(\Lambda) \) is a character on the Lie algebra \( \mathfrak{h}_+(\Lambda), e = \sum d_\alpha e_\alpha \in \mathfrak{g}_0(\Lambda) \). Here it is assumed that the underlying elements \( f, e \) are different from 0, and the coefficients \( c_\alpha, d_\alpha \) of the root vectors \( e_\alpha \), \( e_\alpha \), corresponding to odd roots \( \alpha \) in the system of simple roots \( \Pi \), can be included in a system of generators of the algebra \( \Lambda \). The equations of motion have Laxian form \( \Lambda = [L, L_\epsilon] \).

Here "+" denotes projection onto \( \mathbb{N}_+(\Lambda) \).

2. For \( \mathfrak{sl}(m|n), \text{osp}(2m|l|n) \) and \( \text{osp}(2m+1|l|n) \) each system of simple roots is defined by some subset \( \tau \) in the set of indices \( I = \{1, \ldots, m+n\} \). We set \( p(i) = \overline{0} \) if \( i \in I \setminus \tau \), \( p(i) = \overline{1} \) if \( i \in \tau \), \( p(0) = 0 \) and \( p(0) = \overline{2} \). The set \( I \) is thus partitioned into segments consisting of sequences of indices of the same parity. By \( n_\overline{0} \) (respectively, \( n_\overline{1} \)) we denote the length of the \( i \)-th "even," respectively \( j \)-th "odd" segment.

THEOREM 2.1. The Hamiltonian systems listed in Table 1 are completely integrable. The integrals of motion have the form \( \text{tr} L^k \) for \( \mathfrak{g}(n) \), \( \text{str} L^k \) for \( \mathfrak{sl}(m|n), \text{osp}(m|l|n) \).

The solutions are given by the formulas from Table 2. The numbering in Table 2 corresponds to the numbering in Table 1.

COROLLARY 2.1. Let \( \lambda_1, \ldots, \lambda_k \) be the eigenvalues of the matrix \( \Lambda(0) \). For \( \mathfrak{sl}(m|n) \) and \( \text{osp}(m|l|n) \) they are even elements of the Grassman algebra, for \( \mathfrak{g}(n) \) they are arbitrary. Then the solutions for the superized Toda lattices are rational expressions in \( \exp t \lambda_1, \ldots, \exp t \lambda_k \).

Remark. By eigenvalues of the matrix \( \Lambda \) we mean elements (they are defined nonuniquely) of the diagonal matrix \( \mathfrak{d} \), to which one can reduce the matrix \( \Lambda \), if all the eigenvalues of the underlying matrix \( \mathfrak{A} \) are distinct: \( \Lambda = C^{-1} \mathfrak{d} C \).

It is evident from the formulas given in Table 2 that one can get analytic expressions for the solutions if the elements of the matrix \( \exp t \Lambda(0) \) are calculated explicitly. One can show that the calculation of the latter reduces to the calculation of the block-diagonal matrix \( \exp t \Lambda(0) \) which can be calculated if its blocks have order 2 or 1.