It is shown that if the $\Lambda - \mathcal{Q}$ -bimodule $M$ generates a category of $\Lambda - \mathcal{Q}$ -bimodules, then the ideal of identities of the triangular extension of the direct sum of algebras $\Lambda$ and $\mathcal{Q}$ by means of the bimodule $M$ is equal to the product of ideals of identities of the algebras $\Lambda$ and $\mathcal{Q}$.

In what follows, "algebra" means an associative algebra with identity over a fixed field $F$. The totality $T(A)$ of polynomials of the free algebra $F[X] = F[x_1, \ldots, x_n]$, which are identities of the algebra $A$, is a completely characteristic ideal (a T-ideal) of the algebra $F[X]$ and is called the ideal of identities of the algebra $A$. This paper is devoted to the study of identities of the algebra obtained by a triangular extension of the direct sum of two algebras by means of a certain bimodule. To be precise, let $\Lambda$ and $\mathcal{Q}$ be algebras, and let $M$ be a $(\Lambda - \mathcal{Q})$ -bimodule. We consider the algebra of triangular matrices

$$
\begin{pmatrix}
\Lambda & M \\
0 & \mathcal{Q}
\end{pmatrix} = \left\{ \begin{pmatrix}
\lambda & \mu \\
0 & \omega
\end{pmatrix} \right\}_{\lambda \in \Lambda, \mu \in M, \omega \in \mathcal{Q}}.
$$

The aim of the paper is to find out conditions which ensure that the following equality between ideals of identities holds:

$$T\left( \begin{pmatrix}
\Lambda & M \\
0 & \mathcal{Q}
\end{pmatrix} \right) = T(\Lambda) \cdot T(\mathcal{Q}). \quad (1)$$

Certain general approaches, based on the methods of homological algebra and the theory of representations of algebras, are suggested for investigating this problem. In particular, it will be shown how, with the help of these methods, one can derive certain results of [1, 3, and 4].

**Theorem 1** (Levin [4]). Let $\Lambda = F[X]/T_1, \mathcal{Q} = F[X]/T_2$, where $T_1$ and $T_2$ are T-ideals of the algebra $F[X]$, and let $M$ be a free $(\Lambda - \mathcal{Q})$ -bimodule with one generating element. Then equality (1) holds.

We give a proof of this theorem which uses the arguments of homological algebra.

First of all, we note that, for arbitrary algebras $\Lambda$ and $\mathcal{Q}$, one has the following result.

**Lemma 1.** If $N_1$ is a submodule or a factor of a $(\Lambda - \mathcal{Q})$ -bimodule $N$ then one has the inclusion $T\left( \begin{pmatrix}
\Lambda & N_1 \\
0 & \mathcal{Q}
\end{pmatrix} \right) \supseteq T\left( \begin{pmatrix}
\Lambda & N \\
0 & \mathcal{Q}
\end{pmatrix} \right)$. If $N_1$ is the direct product or the direct sum of some set of copies of the $(\Lambda - \mathcal{Q})$ -bimodule $N$, then one has the equality $T\left( \begin{pmatrix}
\Lambda & N_1 \\
0 & \mathcal{Q}
\end{pmatrix} \right) = T\left( \begin{pmatrix}
\Lambda & N \\
0 & \mathcal{Q}
\end{pmatrix} \right)$.

The proof of Lemma 1 is evident.

Suppose that the conditions of Theorem 1 are satisfied.

LEMMA 2. If, for some \((\Lambda-\Omega)\) -bimodule \(\mathcal{N}\), one has the \(T\)-ideal inclusion
\[ T((\Lambda^N_0)) \subset T(\Lambda)T(\Omega), \]
then equality (1) holds.

In fact, the module \(\mathcal{N}\) is a factor module of the direct sum of some set \(I\) of copies of the module \(\mathcal{M}\), and so, by Lemma 1,
\[ T((\Lambda M_0)) \supset T((\Lambda M_0)) \subset T(\Lambda)T(\Omega). \]
The converse inclusion is evident.

It is particularly easy to complete the proof of Theorem 1 if \(T_1 = T_2 = T\). In fact, in this case, we have the following extension of the algebra \(\Lambda\) with the kernel \(T/T^2\):
\[ 0 \rightarrow T/T^2 \rightarrow F[X]/T^2 \rightarrow \Lambda \rightarrow 0. \]

The set of classes of equivalent extensions of the algebra \(\Lambda\) with the kernel \(T/T^2\) is in one-to-one correspondence with the group \(H^2(\Lambda, T/T^2)\). We imbed the module \(T/T^2\) in the injective \((\Lambda-\Omega)\) -bimodule \(\mathcal{N}: T/T^2 \rightarrow \mathcal{N}\). This imbedding induces a group homomorphism \(H^2(\Lambda, T/T^2) \rightarrow H^2(\Lambda, \mathcal{N})\) which takes an element of \(H^2(\Lambda, T/T^2)\), corresponding to the extension (2), into the element corresponding to the extension \(0 \rightarrow \mathcal{N} \rightarrow \Lambda \rightarrow 0\) of the algebra \(\Lambda\) with the kernel \(\mathcal{N}\). In addition, there exists an imbedding \(i_z: F[X]/T^z \rightarrow \Lambda\), for which the following diagram is commutative:

\[
\begin{array}{cccccc}
0 & \rightarrow & T/T^z & \rightarrow & F[X]/T^z & \rightarrow & \Lambda & \rightarrow & 0 \\
\downarrow i^z & & \downarrow i_z & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{N} & \rightarrow & \Delta & \rightarrow & \Lambda & \rightarrow & 0
\end{array}
\]

Since, for an injective module \(\mathcal{N}\), the group \(H^2(\Lambda, \mathcal{N})\) is null, therefore, the lower extension in this diagram can be decomposed: \(\Delta = \mathcal{N} + \Lambda\). Associating with the element \(\psi + \lambda\) \((\psi \in \mathcal{N}, \lambda \in \Lambda)\) the matrix \((\Lambda^M_0)_{(0, \lambda)} \in (\Lambda^N_0, \Lambda)\), we get the algebra imbedding \(\Delta \rightarrow (\Lambda^N_0, \Lambda)\). Therefore \(T((\Lambda^N_0)) \subset T(\Lambda) \subset T(F[X]/T^z) - T^z\). One may now refer to Lemma 2.

The general case is similar to that considered above, but is technically slightly more complicated. Let \(\Sigma = F[X]/(T_1 T_2), \Gamma = F[X]/T_1 T_2\), and \(K = (T_1 T_2)/T_1 T_2\). We have the following natural extension of the algebra \(\Sigma\) with the kernel \(K\):
\[ 0 \rightarrow K \rightarrow \Gamma \rightarrow \Sigma \rightarrow 0. \]

The kernel \(K\) of this extension has the property that \(T_i K = K T_i = 0\) and can, therefore, be regarded as a \((\Lambda-\Omega)\) -bimodule. As in [2], Chapter XIV, one can show that the extensions of the algebra \(\Sigma\) with the kernel \(K\) are in one-to-one correspondence with the elements of the group
\[ \text{Ext}^2_{\Sigma/\Sigma + \Sigma T_2}(\Sigma/(T_1 \Sigma + \Sigma T_2), K) = \text{Ext}^2_{\Sigma/\Sigma + \Sigma T_2}(\Sigma/(T_1 \Sigma + \Sigma T_2), K). \]

We imbed the module \(K\) in the injective \((\Lambda-\Omega)\) -bimodule \(\mathcal{N}: \mathcal{N} \rightarrow \Sigma\). On constructing, corresponding to (3), the extension \(0 \rightarrow \mathcal{N} \rightarrow \Sigma \rightarrow 0\) of the algebra \(\Sigma\) with the kernel \(\mathcal{N}\) and the imbedding \(j_z: \Gamma \rightarrow \Delta\), we get the commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & \Gamma & \rightarrow & \Sigma & \rightarrow & 0 \\
\downarrow i^z & & \downarrow i_z & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{N} & \rightarrow & \Delta & \rightarrow & \Sigma & \rightarrow & 0
\end{array}
\]

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