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HIGHER REGULATORS AND VALUES OF L-FUNCTIONS

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In the work conjectures are formulated regarding the value of L-functions of motives and some computations are presented corroborating them.

INTRODUCTION

Let X be a complex algebraic manifold, and let $K_j(X)$, $H^j_{\mathcal{B}}(X,\mathbf{Q})$ be its algebraic K-groups and singulary cohomology, respectively. We consider the Chern character ch: $K_j(X)\otimes\mathbf{Q}\to\oplus H^{2i-j}_{\mathcal{B}}(X,\mathbf{Q})$. It is easy to see that there are the Hodge conditions on the image of ch: we have ch $(K_j(X))\subset\oplus(W_{2i}H^{2i-j}_{\mathcal{B}}(X,\ \mathbf{Q}))\cap(F^iH^{2i-j}_{\mathcal{B}}(X,\mathbf{C}))$, where $W_.$, $F^{\cdot}$ are the filtration giving the mixed Hodge structure on $H^{\cdot}_{\mathcal{B}}(X)$. For example, if X is compact, then ch $(K_j(X)) = 0$ for $j > 0$. It turns out that the Hodge conditions can be used, and, untangling them, it is possible to obtain finer analytic invariants of the elements of $K_.(X)$ than the usual cohomology classes. For the case of Chow groups they are well known: they are the Abel–Jacobi–Griffiths periods of an algebraic cycle. Apparently, these invariants are closely related to the values of L-functions; we formulate conjectures and some computations corroborating them.

In Sec. 1 our main tool appears: the groups $H^j_{\mathcal{D}}(x,\mathbf{Z}(i))$ of "topological cycles lying in the i-th term of the Hodge filtration." These groups are written in a long exact sequence

$$\ldots\to H^{j-1}_{\mathcal{B}}(X,\mathbf{C})\to H^j_{\mathcal{D}}(X,\mathbf{Z}(i))\xrightarrow{\varepsilon_{\mathcal{B}}\oplus\varepsilon_{\mathcal{F}}} H^j_{\mathcal{B}}(X,\mathbf{Z})\oplus F^iH^j_{\mathcal{B}}(X,C)\to\ldots.$$

On $H^{\cdot}_{\mathcal{D}}$ we construct a $\cup$-product such that $\varepsilon_{\mathcal{B}}$ becomes a ring morphism, and we show that $H^{\cdot}_{\mathcal{D}}$ form a cohomology theory satisfying Poincaré duality. Therefore, it is possible to apply the machinery of characteristic classes to $H^{\cdot}_{\mathcal{D}}$ [22] and obtain a morphism $\mathrm{ch}_{\mathcal{D}}:K_j(X)\otimes\mathbf{Q}\to\oplus H^{2i-j}_{\mathcal{D}}(X,\ \mathbf{Q}(i))$. The corresponding constructions are recalled in Sec. 2. Let $H^{2i-j}_{\mathcal{A}}(X,\ \mathbf{Q}\ (i))\subset K_j(X)\otimes\mathbf{Q}$ be the eigenspace of weight i relative to the Adams operator [2]; then $\mathrm{ch}_{\mathcal{D}}$ defines a regulator – a morphism $r_{\mathcal{D}}:H^j_{\mathcal{A}}(X,\mathbf{Q}(i))\to H^j_{\mathcal{D}}(X,\mathbf{Q}(i))$. [It is thought that for any schemes there exists a universal cohomology theory $H^j_{\mathcal{A}}(X,\ \mathbf{Z}(i))$, satisfying Poincaré duality and related to Quillen's K-theory in the same way as in topology the singular cohomology is related to K-theory; $H^{\cdot}_{\mathcal{A}}$ must be closely connected with the Milnor ring.] In the appendix we study the connection between deformations of $\mathrm{ch}_{\mathcal{D}}$ and Lie algebra cohomologies; as a consequence we see that if X is a point, then our regulators coincide with Borel regulators. There we present a formulation of a remarkable theory of Tsygan–Feigin regarding stable cohomologies of algebras of flows. Finally, Sec. 3 contains formulations of the basic conjectures connecting regulators with the values of L-functions at integral points distinct from the middle of the critical strip; the arithmetic intersection index defined in part 2.5 is responsible for the behavior in the middle of the critical strip. From these conjectures (more precisely, from the part of them that can be applied to any complex manifold) there follow rather unexpected assertions regarding the connection of Hodge structures with algebraic cycles. The remainder of the work contains computations corroborating the conjectures in Sec. 3. Thus, in Sec. 7 we prove these conjectures for the case of Dirichlet series;

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Sec. 5 contains a result giving a partial proof of the conjecture for values at two of L-functions of curves uniformized by modular functions; Sec. 6 contains an analogous computation for the product of curves of this type.

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NOTATION

We shall use the standard language of cohomological algebra. If A is an Abelian category, then $\mathbf{D}(A)$ is the derived category of A; $\mathbf{D}F(A)$ is the filtered derived category; C(A) is the category of complexes; s is a functor assigning to a bicomplex the corresponding simple complex; $N: A^{\Delta} \to C(A)$ is the normalization functor (A^{Δ} are the cosimplicial objects of A). If $X^{\cdot}$ is a complex, then the complex $X^{\geqslant i}$ coincides with $X^{\cdot}$ in degrees $\geqslant i$ and is equal to zero in degrees $<i$ (the i-th term of the filtration group on X). We denote by $[Y^0 \to Y^1 \to Y^2 \to \dots]$ the complex equal to zero in negative degrees and coinciding with $Y^{\cdot}$ in positive degrees. If T is the topology in $\mathcal{F}(T)$ — the category of sheaves of Abelian groups on T, then $C(T) := C(\mathcal{F}(T))$.

"An analytic space" is an analytic space over $\mathbf{R}$; we denote by $\mathcal{A}n$ the category of analytic spaces equipped with the usual topology.

Let $V \in \mathcal{A}n$. Then a sheaf $\mathcal{F}$ on V is a sheaf $\mathcal{F}_{\mathbf{C}}$ on $V(\mathbf{C})$ equipped with the action of an involution σ of complex conjugation on $V(\mathbf{C})$; the spectral sequence with second term $H^p(\mathbf{Z}/2, H^q(V(\mathbf{C}), \mathcal{F}_{\mathbf{C}}))$ converges to the cohomologies $H^{\cdot}(V, \mathcal{F})$ (the Leray sequence of the structural morphism $V \to \operatorname{Spec} \mathbf{R}$); in particular, for a $\mathbf{Q}$-sheaf we have $H^{\cdot}(V, \mathcal{F}) = H^{\cdot}(V(\mathbf{C}), \mathcal{F}_{\mathbf{C}})^{\sigma}$.

We denote by $\mathbf{C}_V$ or simply $\mathbf{C}$ the local system on V corresponding to a constant sheaf with stalk $\mathbf{C}$ on $V(\mathbf{C})$ with the action σ by means of complex conjugation. Identifying $\mathbf{C}_V$ with the subsheaf of constant functions in the structural sheaf $\mathcal{O}_V$, we obtain, if V is smooth, the isomorphism $H^{\cdot}(V, \mathbf{C}) = H^{\cdot}_{\mathcal{D}\mathcal{R}}(V)$. If $K \subset \mathbf{C}$ is a subgroup closed relative to conjugation, then let K_V (or simply K) be the subsheaf of $\mathbf{C}_V$ with stalk K. We need the following subgroups of this kind: for a subring $A \subset \mathbf{R}$ and $n \in \mathbf{Z}$ we set $A(n) := (2\pi i)^n A = A \cdot \mathbf{Z}(n) \subset \mathbf{C}$; then $A(i) \cdot A(j) = A(i+j)$. If $\mathcal{F}$ is a sheaf on V, then let $\mathcal{F}(n) := \mathcal{F} \otimes \mathbf{Z}(n)$; it is clear that for a sheaf of $\mathbf{C}_V$-modules $\mathcal{F}$ the sheaf $\mathcal{F}(n)$ can be canonically identified with $\mathcal{F}$.

If V is smooth, then let $\mathcal{O}_{V\infty} \supset \mathcal{O}_V$ be the sheaf of functions of class C^{∞} on V, let $\Omega^{\cdot}_{V\infty} \supset \Omega^{\cdot}_V$ be the corresponding de Rham complex, and let $\Omega^{n}_{V\infty} = \bigoplus_{p+q=n} \Omega^{p,q}_{V\infty}$; let $\mathbf{S}^{\cdot}_V$ be the subsheaf of $\mathbf{R}$-valued forms, $\Omega^{\cdot}_{V\infty} = \mathbf{S}^{\cdot}_V \otimes_{\mathbf{R}} \mathbf{C}_V$. If X is an algebraic manifold over $\mathbf{R}$, then we set $H^{\cdot}_{\mathcal{B}}(X, A(n)) = H^{\cdot}(X_{\text{an}}, A(n))$. For any cohomology theory $H^{\cdot}_?$ we denote by $H^{\cdot}_?{}^{\cdot}$ the corresponding cohomology groups (see 2.3).

CHAPTER 1. MAIN CONSTRUCTIONS AND CONJECTURES

1. $\mathcal{D}$-Cohomologies

1.1. $\mathcal{D}$-Cohomologies of Analytic Spaces.

We fix a subring $A \subset \mathbf{R}$.

Definition 1.1.1.[1] For $i \in \mathbf{Z}$ we define a complex $A(i)_{\mathcal{D}}$ of sheaves on $\mathcal{A}n$ by the formula

$$A(i)_{\mathcal{D}} := \operatorname{Cone}(F^i \oplus A(i) \to \Omega^{\cdot})[-1].$$

Here $\Omega^{\cdot}$ is the de Rham complex of holomorphic forms equipped with the filtration group $F^j := \Omega^{\geqslant j}$; the arrow $F^i \oplus A(i) \to \Omega^{\cdot}$ is the difference of the obvious imbeddings $F^i \hookrightarrow \Omega^{\cdot}$ and $A(i) \hookrightarrow \mathbf{C} \hookrightarrow \Omega^{\cdot}$. ■

Let ε_F, ε_A be the natural morphisms of $A(i)_{\mathcal{D}}$ into F(i), A(i), respectively. We have the exact triangle

$$\dots \to \Omega^{\cdot}[-1]^{\alpha} \to A(i)_{\mathcal{D}} \xrightarrow{\varepsilon_F + \varepsilon_A} F(i) \oplus A(i) \to \dots \qquad (*)$$

[1] Apparently, these complexes were first considered by Deligne (see [8]).