FRactal STRAIGHT LINES AND QUASISYMMETRIES

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Introduction

0.1. The subject of research in the present article is the lines in Euclidean space which are, in some sense, "straight to within $\varepsilon$." Examining whether a line in a physical problem is straight, an experimenter is limited by the precision of the measuring instrument and the method of measurement. It is natural to suppose that the error is proportional to the size of the object under study. Such approach leads readily to nonsmooth curves. A differentiable curve resembles a straight line more and more strongly as an observable arc becomes shorter. Such an arc can be placed in a right circular cylinder with the ratio of the radius to the length tending to zero as the length of the cylinder vanishes.

By a fractal straight line we mean a curve for which this ratio does not tend to zero but remains bounded above by a small $\varepsilon > 0$ uniformly in all arcs of the line (see a definition in Subsection 1.2). Another approach gives the same classes of curves. Imagine that we have all possible photographs of an object, with the sizes of the photographs all the same and equal to $d \times d$, but we do not know which photograph corresponds to the given area nor the scale of magnification. Assume that in all the photographs the object looks like a slightly indistinct almost rectangular strip of width at most $de$, where $\varepsilon$ is small. Assuming that the sharpness of the image is constant, depends only on the photographic camera, and do not depend on the sizes of the portrayed fragment, we must arrive at the conclusion that the object is a fractal straight line. In a real physical situation, we can simulate such an object by means of fractal straight lines within certain ranges for scales.

Not giving a precise definition of fractal objects, we point out that at an early stage of their study [1] it was self-similarity that was taken as one of their defining properties: separate parts of a fractal set are similar to each other. Here the similarity of objects can be interpreted in different ways: as geometric similarity (i.e. existence of a mapping changing all distances at a fixed ratio), statistical similarity, or similarity in the sense of membership of objects in a certain class invariant under the mappings preserving the ratio of distances.

0.2. While examining stability questions for quasiconformal mappings, the author introduced classes of mappings called $h$-similarities (see a definition in 1.4). For the mappings of the Euclidean space $R^n$ with $n \geq 2$ into itself, these classes are equivalent to the classes of $K$-quasiconformal mappings with $K$ sufficiently close to unity. Over a rather broad class of subsets of $R^n$, in particular on straight lines, these mappings are quasisymmetries (see Subsection 1.4). Description for images of a straight line under the mappings is one of the main goals in studies of classes of mappings. It is unbounded quasicircles that are well known; these are curves in the plane satisfying the Ahlfors $M$-condition [2]. They are the images of straight lines under quasiconformal mappings of the extended complex plane each of which takes the point at infinity to itself. In the space, the Ahlfors condition is necessary but in general not sufficient for a curve to be the image of a straight line under a quasiconformal mapping of $R^n$. There are topological constraints: at large $M$ the $M$-condition does not prevent the curve knotting. The $M$-condition is equivalent to the condition $J_2(\varepsilon)$, which the author introduces below, but only at sufficiently large $\varepsilon$. In our problems, the shortcoming of the $M$-condition, $M \geq 1$, is the fact that at $M = 1$ it does not make straight lines into a special class. The curves satisfying the $M$-condition for every $M \geq 1$ may have angles up to $\pi/2$.

0.3. The principal result of the present article is a theorem claiming existence of an $h$-similarity,

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i.e. a quasisymmetric mapping, of a Euclidean straight line onto a fractal straight line. Afterwards, following the plan exposed in [3], we extend the mapping to a quasisymmetric mapping of the Euclidean space into itself, while preserving the order of proximity of the mapping to a similarity. The converse assertion is valid as well: the image of a straight line under a K-quasiconformal mapping of the space with K sufficiently close to unity is a fractal straight line. Thus, we acquire a convenient tool for studying fractal straight lines. It is interesting to study the measures that are induced on fractal straight lines by such mappings. In turn, the possibility of being applied to a series of physical problems provides additional stimuli to the theories of spatial quasiconformal mappings, quasisymmetries, and quasi-isometries.

0.4. It is worth adding that, while studying spatial quasiconformal and quasi-isometric mappings, the author revealed one more series of fractal objects. In [4] there were considered fractal hyperplanes, the surfaces that are “flat to within ε”; i.e., inside each ball of radius r, such a surface lies at a distance at most εr from the intersection of the ball and a hyperplane. It turns out that, for a surface to be a fractal hyperplane at a small ε, it is necessary and sufficient that it admit a quasi-isometric reflection of the space which is close to an isometry.

In [5] Mandelbrot introduced the notion of self-affine fractals. In contradistinction to self-similarity, here we must distinguish some directions and some affine group that preserves these directions. In [6] the author defined classes of mappings “quasipreserving cones.” These mappings, being quasiconformal, carry parallel straight lines, the axes of cones, into the class of Lipschitz curves (a narrower class of fractal straight lines).

1. Preliminaries and Statement of Results

1.1. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. We identify the real line $\mathbb{R}$ with the coordinate axis $\mathbb{R} = (x_1, 0, ..., 0) \subset \mathbb{R}^n$. The completion of $\mathbb{R}^n$ with some point at infinity is denoted by $\overline{\mathbb{R}^n}$. The symbols $B(x, r)$ and $S(x, r)$ denote the open ball and sphere with center $x$ and radius $r$, and $\overline{B(x, r)}$ is the closed ball. We let $[x, y]$ denote not only an interval of the real line but also a segment with endpoints $x$ and $y$ taken arbitrarily in the space $\mathbb{R}^n$. As usual, $C$ stands for the complex plane; $\mathbb{C}$ is the extended plane. We define $\rho(x, A) = \inf_{y \in A} |x - y|$, the distance from a point $x$ to a set $A$, and $\dist(A, B) = \sup_{x \in A} \rho(x, B)$, the deviation of a set $A$ from a set $B$. Observe that the Hausdorff distance between the sets equals $\max\{\dist(A, B), \dist(B, A)\}$; however, in the cases when $B$ is a segment of a straight line with endpoints $a$ and $b$ and $A$ is a continuous curve with the same endpoints, the deviation $\dist(A, B)$ coincides with the Hausdorff distance.

An essential role in further constructions is played by Möbius transformations of $\overline{\mathbb{R}^n}$, i.e. the compositions of finitely many isometries and inversions in spheres, redefined at the points $\infty$ and $f^{-1}(\infty)$. We denote the group of Möbius transformations of $\overline{\mathbb{R}^n}$ by $M_n$. A Möbius transformation of $\overline{\mathbb{R}^n}$ is a similarity if it takes $\infty$ to $\infty$. We denote the group of similarities by $\Pi_n$. If $T \in \Pi_n$ then $\|T\|$ denotes the dilation coefficient of $T$ which equals $|T(x) - T(y)|/|x - y|$ for any pair of points $x, y \in \mathbb{R}^n$, $x \neq y$.

1.2. In the present article we study unbounded curves. Here by an unbounded curve $\gamma$ we mean a parametrized curve which is given by a continuous mapping $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\gamma(\infty) = \infty$. The symbol $\gamma$ will also denote $\gamma(\overline{\mathbb{R}})$, the image of the real line under $\gamma$.

**DEFINITION.** We say that a curve $\gamma$ satisfies condition $J_1(\varepsilon)$ if for all $x, y \in \mathbb{R}$

$$\dist(\gamma([x, y]), [\gamma(x), \gamma(y)]) \leq \varepsilon |\gamma(x) - \gamma(y)|.$$  

We say that $\gamma$ satisfies condition $J_2(\varepsilon)$ if, for all $x, y, z \in \mathbb{R}$, the inequalities $x \leq y \leq z$ imply that

$$(1 + \varepsilon^2)|\gamma(x) - \gamma(z)| \leq |\gamma(x) - \gamma(y)| + |\gamma(y) - \gamma(z)|.$$  

Obviously, the classes of curves satisfying conditions $J_1(\varepsilon)$ and $J_2(\varepsilon)$ are invariant under the similarity transformations.