We consider the class $\Pi$ of operator functions $f(z)$, meromorphic for $|z| \neq 1$, which admit a representation in the form $f(z) = f_2^{-1}(z)f_1(z)$, where $f_1(z)$ is an operator-valued holomorphic function and $f_2(z)$ a scalar-valued bounded holomorphic function, such that the strong limits $\lim_{z \to \zeta} f_1(z)$ and $\lim_{z \to \zeta} f_2(z)$ coincide a.e. on the unit circle $|\zeta| = 1$. We prove that any function of class $\Pi$ can be represented as a block of some $J$-inner function of class $\Pi$. We describe all such representations. The results found are applied to the question of realizations of functions of class $\Pi$ as transfer functions of linear systems.

1. Formulation of the Basic Results

The results recounted in this paper were mainly found by the author in 1975 in the course of work on the realizations of passive systems with loss by the method of Darlington [1]. Their publication was delayed for a variety of reasons.

We shall consider not only scalar-valued and matrix-valued but also operator-valued functions, taking values in $[M, N]$, which does not evoke any additional difficulties. As usual, we denote by $[M, N]$ the set of bounded linear operators from the space $M$ to $N$. All the spaces considered are assumed to be separable Hilbert spaces.

At the center of our attention will be the class $\Pi$ of functions $f(z)$, meromorphic for $|z| \neq 1$, with values in $[M, N]$, for which two conditions hold: 1) they can be represented as ratios

$$f(z) = \frac{f_2^{-1}(z)f_1(z)}{f_2(z)}$$

of bounded functions which are holomorphic for $|z| \neq 1$, where $f_1(z)$ is operator-valued and $f_2(z)$ is scalar-valued, that is, they have Nevanlinna characteristic which is bounded in the uniform sense for $|z| \neq 1$, and 2) almost everywhere (a.e.) on the unit circle $|\zeta| = 1$,

$$\hat{f}(\zeta) = \hat{f}_1(\zeta)e^{i\theta(\zeta)} - \hat{f}_2(\zeta) = e^{i\theta(\zeta)}(\hat{f}(\zeta) - \hat{f}_1(\zeta))$$

The class $\Pi$ is defined analogously for meromorphic functions $f(z)$ for $\Re z \neq 0$ (or $\Re z = 0$).

Functions of class $\Pi$ arose previously in various investigations: the approximation of meromorphic functions by rational ones [2], cyclic vectors of the operator of left translation in the space $L^2$ [3], factorization of nonnegative functions [4], the Sturm-Liouville boundary problem [5], the realization of linear systems [6].

If $f(z)$ is a rational function, then as is easy to see, it belongs to the class $\Pi$ both for $|z| \neq 1$ and for $\Re z \neq 0$ (or $\Re z = 0$). If $f(z)$ is an entire scalar function, then, as Krein [5] proved, for $\Re z \neq 0$ it belongs to the class $\Pi$ if and only if

$$\left(\begin{array}{c} 0 \\ \infty \end{array}\right), \quad \left(\begin{array}{c} \infty \\ 0 \end{array}\right)$$

that is, when $f(z)$ is of finite degree and Cartright class. It is obvious that the set of scalar functions of class $\Pi$ is a field with the usual definitions of the operations of addition and multiplication of functions. Thus one already has a wide possibility of constructing examples of functions of class $\Pi$. In what follows we shall indicate a method of constructing functions of class $\Pi$ which is important for us.
Let \( f \) be a self-adjoint unitary operator: \( f^* f = f f^* = I \). The function \( f(z) \) is said to be bilaterally \( f \)-inner in the disk \( K = \{ z : |z| < 1 \} \), if it: 1) is meromorphic in \( K \), 2) at each point of \( K \) where it is holomorphic, assumes bilaterally \( f \)-contracting values, that is,
\[
\text{if } |f(z)| < 1 \quad \text{and} \quad |f(z)|^2 < 1 \quad (z \in K),
\]
3) almost everywhere on the unit circle has \( f \)-unitary boundary values, that is,
\[
\text{if } |f(z)| = 1 \quad \text{and} \quad |f(z)|^2 = 1 \quad (z \in \partial K).
\]

For \( f = I \) such a function is said to be bilaterally inner. One defines bilaterally \( f \)-inner functions in the half planes \( \text{Im} \, z > 0 \) and \( \text{Re} \, z > 0 \) analogously. If the bilaterally \( f \)-inner function \( f(z) \) assumes values in \([\mathcal{N}, \mathcal{N}]\) and \( \dim \mathcal{N} < \infty \), then \( f(z) \in \mathcal{N} \). In fact, the simple formulas
\[
\begin{align*}
\text{if } |f(z)| < 1 \quad \text{and} \quad |f(z)|^2 < 1, \\
p = \frac{1}{2} (I + f), \\
q = \frac{1}{2} (I - f),
\end{align*}
\]
are known [10, 7], which establish a connection between bilaterally \( f \)-contracting functions \( f(z) \) which are meromorphic in \( K \), and functions \( s(z) \) of class \( \mathcal{B} \), which are holomorphic in \( K \) with \( \|s(z)\| \leq 1 \). The expression of \( f(z) \) in terms of \( s(z) \) shows that \( f(z) \) can be represented in the form (1) in \( K \). Now if we have (3) here, then we can define \( f(z) \) in \( K_e = \{ z : 1 < |z| \leq +\infty \} \) by the symmetry principle: \( \text{if } |f(z)| < 1 \quad \text{and} \quad |f(z)|^2 < 1 \), then \( f(z) \) can be represented in the form (1) in \( K_e \) also. Moreover, condition (2) will hold, so that \( f(z) \in \mathcal{N} \). Now if \( \dim \mathcal{N} = \infty \), then a bilaterally \( f \)-inner function \( f(z) \) does not necessarily belong to the class \( \mathcal{N} \). For it we will have the inclusion \( f(z) \in \mathcal{N} \) if and only if \( f(z) \) and \( f^{-1}(z) \) can be represented in \( K \) in the form (1), that is, when \( f^{-1}(z) \) and \( f(z) \) have a scalar multiple in the sense of [11].

Bilaterally \( f \)-contracting functions \( f(z) \) which are meromorphic in \( K \) were the subject of the investigation of Potapov [7], and later that of Ginzburg (for \( \dim \mathcal{N} = \infty \)) [13]. Rational bilaterally \( f \)-inner functions are finite products of elementary Blaschke–Potapov factors. Infinite convergent Blaschke–Potapov products for \( \dim \mathcal{N} < \infty \) are also bilaterally \( f \)-inner functions [8]. Rational real bilaterally \( f \)-inner matrix-functions \( f(z) \) are transmission matrices of circuit data (for \( f = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)) or dissipation of data (for \( f = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \)) of finite ideal linear passive electrical circuits without loss (without resistance) [9]. The monodromy matrix \( W(z) = W(\ell, z) \) of an arbitrary self-adjoint regular canonical differential system
\[
\frac{dW}{dx} = -z f H(x) W, \quad 0 < x < \ell \quad (W(0, z) = I),
\]
\[
(H(x) > 0, \quad \int_0^\ell \|H(x)\| \, dx < \infty)
\]
is an entire bilaterally \( f \)-inner function for \( \text{Re} \, z > 0 \) with \( W(0) = I \), as is easily verified. In the finite-dimensional case (\( \dim \mathcal{N} < \infty \)) by a theorem of Potapov [7] the converse is also true: An arbitrary entire function \( W(z) \) which is bilaterally \( f \)-inner for \( \text{Re} \, z > 0 \) with \( W(0) = I \) is the monodromy matrix of some self-adjoint regular canonical system (5). Now if \( W(z) \) is real in addition, then this is the transmission matrix of circuit data (for \( f = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)) or dissipation of data (for \( f = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \)) of a regular ideal n-conductive line without loss with