Now we take \( k = n + 1 \), and \( \{ \lambda_i : 1 \leq i \leq n + 1 \} \) to be the set of points of \( \Lambda \) lying closest to \( \lambda - \lambda_{n+2} \in \Lambda' \); then one of them (say \( \lambda_1 \)) does not occur in the set \( \mathcal{M}_{n+2} \), and consequently
\[
\Delta^{m+1} w(\lambda^{m+1}) = w(\lambda) \prod_{j=1}^{m+1} \zeta_j = \zeta_j.
\]

But since for any \( \varepsilon > 0 \) one can find \( \lambda_{n+2} \in \Lambda' \), for which all points \( \lambda_i, i \leq n + 1 \) fall in \( \mathcal{M}_\varepsilon(\lambda_{n+2}) \), one has \( \sup |\Delta^{m+1} w(\lambda^{m+1})| = \infty \), i.e., \( w \notin \mathcal{X}_{n+1} \).

Thus, we have proved that \( \Lambda \notin \mathcal{U}_{n+1} \mathbb{R} \), and it remains to prove that a sparse subset of \( \Lambda \) is Carleson. By Carleson's theorem it suffices to verify that any bounded function on this subset can be interpolated by functions from \( H^\infty \). For this we extend an arbitrary bounded function from a given sparse subset to all of \( \Lambda \) so that the function \( w \) obtained belongs to the class \( \mathcal{X}_{n+1}(\Lambda) \), and then the condition \( \mathcal{J}_\lambda H^\infty = \mathcal{X}_{n+1}(\Lambda) \) guarantees the possibility of the interpolation required.

Let \( \Lambda' \) be a sparse subset of \( \Lambda \), \( w \in \ell^\infty(\Lambda') \). Since \( \Lambda \) is the union of \( n + 1 \) sparse sets, one can find an \( \varepsilon > 0 \) such that each connected component of the set \( \Lambda_\varepsilon = U \{ \mathcal{M}_\varepsilon(\lambda) : \lambda \in \Lambda \} \) contains no more than \( n + 1 \) points of \( \Lambda \) and no more than one point of \( \Lambda' \). We define \( w \) at those points \( \lambda \in \Lambda \), whose component of \( \Lambda_\varepsilon \) does not contain points of \( \Lambda' \), to be zero, and we set \( w(\lambda) = w(\lambda') \), if \( \lambda \in \Lambda' \) and \( \lambda \) lies in the same component of \( \Lambda_\varepsilon \) as \( \lambda' \). As already noted to verify the boundedness of the divided differences, one need not consider distant points, so it suffices to verify the boundedness of those differences \( \Delta^k w(\lambda^{(k+1)}) \) for which all points \( \lambda_i \) lie in one component of \( \Lambda_\varepsilon \), but by definition of the function \( w \) all the first and hence all subsequent differences for such points are equal to zero. Thus \( \Lambda' \in C \), i.e.,
\[
\mathcal{J}_\lambda H^\infty = \mathcal{X}_{n+1}(\Lambda) \Rightarrow \Lambda \notin \mathcal{U}_{n+1} \mathbb{R}.
\]

LITERATURE CITED


MULTIPLIERS ON BESOV SPACES

A. B. Gulisashvili

It is proved in this paper that the characteristic function of the half-space is not a multiplier for the pair \( (B^{1/p}_p, B^{1/q}_q) \), \( 1 < p < \infty, 1 < q \leq \infty \). In addition, necessary and sufficient conditions are found for the validity of the inclusion
\[
\mathcal{X}_{\varepsilon} = \mathcal{M}(B^{1/p}_p \rightarrow B^{1/q}_q).
\]

Let \( A_1 \) and \( A_2 \) be function spaces on \( \mathbb{R}^{n+1} \). A function \( f \), defined on \( \mathbb{R}^{n+1} \), is called a multiplier for the pair \( (A_1, A_2) \), if
\[
\|f\|_{A_2} \leq c \|g\|_{A_1}
\]
for all \( g \in A_1 \), and the constant \( c \) is independent of \( g \). The set consisting of all multipliers for the pair \( (A_1, A_2) \) will be denoted by \( \mathcal{M}(A_1 \rightarrow A_2) \).

Properties of multipliers for function spaces have been studied in many papers. We note [9] and particularly [3], where there is a survey of the theory of multipliers and an extensive bibliography. One can read about multipliers on Besov spaces in [11].

In this paper we are interested in conditions under which the characteristic function $\chi_E$ of a Lebesgue measurable set $E$, $E \subset \mathbb{R}^n$, is a multiplier for a pair of Besov spaces. Triebel (cf. [11], Remark 2 in Chap. 2.8.7) posed the question, is the characteristic function of the half-space

$$B^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

a multiplier on the Besov space with limit smoothness $B^{1/p^n}_{1+q}$. In this paper it is shown that the answer to this question is negative. Moreover, we prove the following

**THEOREM 4.1** (cf. Sec. 4 below). The function $\gamma_{B^+}$ does not belong to the set $M(B^{1/p^n}_{1+q} \rightarrow B^{1/p^n}_{1+q})$, $1 < p < \infty$, $1 < q < \infty$.

In addition, it is proved in Sec. 4 that if $n = 1$, it follows from $\gamma_E \notin M(B^{1/p^n}_{1+q} \rightarrow B^{1/p^n}_{1+q})$, $1 < p < \infty$, $1 < q < \infty$ that the set $E$ is trivial [we shall call a set $E \subset \mathbb{R}^n$ trivial, if $\mu_n(E) = 0$ or $\mu_n(\mathbb{R}^n \setminus E) = 0$]. Apparently such a theorem is also valid if $n > 1$.

Things are quite different with the inclusion $\gamma_E \in M(B^{1/p^n}_{1+q} \rightarrow B^{1/p^n}_{1+q})$. In this case $\gamma_{\mathbb{R}^n} \in M(B^{1/p^n}_{1+q} \rightarrow B^{1/p^n}_{1+q})$, $1 < p < \infty$, and moreover, the author has found a simple geometric condition on $E$, which guarantees that $\chi_E$ belongs to the set $M(B^{1/p^n}_{1+q} \rightarrow B^{1/p^n}_{1+q})$. It turns out that the theory of sets with finite perimeter (cf. [2, 12]) plays an important role in this problem. One has

**THEOREM B.** In order that $\gamma_E \in M(B^{1/p^n}_{1+q} \rightarrow B^{1/p^n}_{1+q})$ it is necessary and sufficient that

$$\mathcal{H}_{n-1}((\mathcal{F}^* E) \cap B(x, r)) \leq c r^{n-1}, \quad x \in \mathbb{R}^n, \quad 0 < r < 1$$

with constant $c$ independent of $x$ and $r$.

In the formulation of Theorem B, $\mathcal{H}_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, $B(x, r)$ is the open ball in $\mathbb{R}^n$ with center at the point $x$ and radius $r$, and $\mathcal{F}^* E$ is the reduced boundary of $E$, in other words, the set of those points of the boundary of $E$ at which there exists an approximate outer normal. Theorem B is a special case of Theorem 3.1, which is proved below.

We note that various analogs of condition (1) have appeared in the study of imbedding theorems for spaces of smooth functions (cf. [3]) and in the theory of singular integrals on rectifiable curves in $\mathbb{R}^n$. As was shown by David in [6], the analog of (1) is necessary and sufficient for a singular integral operator on a curve $\Gamma$ to be bounded in $L^p(\Gamma)$ for $1 < p < \infty$.

### 1. Preliminary Information

We denote by $\mathbb{R}^n$, $n > 1$ the $n$-dimensional Euclidean space with Lebesgue measure $\mu_n$, and by $\mathcal{H}_{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

The *Besov space* $B_{pq}^s$, $1 < p < \infty$, $1 < q < \infty$, $0 < s < \infty$ (in this paper we consider only these values of the parameters) is defined as follows (cf. [1, 4, 10, 11]): let the function $\Phi$, $\Phi \in \mathcal{S}'(\mathbb{R}^n)$ be such that 1) $\text{supp } \Phi = \{x : \xi^*|\xi| \leq 2\}$; 2) $\Phi(\xi) > 0$ for $\xi^*|\xi| < 2$; 3) $\sum_{k=\infty}^{\infty} \Phi(2^{-k} \xi) = 1$, $\xi = 0$. We denote by $\Phi_{\infty}(-\infty < k < \infty)$ and $\psi$ the function defined by

$$\Phi_{\infty}(\xi) = \psi(2^{-k} \xi), -\infty < k < \infty, \quad \psi(\xi) = 1 - \sum_{k=1}^{\infty} \psi(2^{-k} \xi),$$

where $\mathcal{F}$ is the Fourier transform. Then

$$B^+ = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{pq}^s} < \infty\},$$