One says that an entire function $f$ of finite exponential type belongs to the Cartwright class $C$, if

$$\int_{-\infty}^{\infty} \frac{\log|f(x)|}{1 + x^2} \, dx < \infty.$$ 

Let $N_+(r)(N_-(r))$ denote the number of zeros of the function $f$ in the disk $|z| \leq R$, such that $\text{Re} \, z > 0$ ($\text{Re} \, z < 0$, respectively). We give a simple derivation of the following result of importance in the theory of entire functions, from a weak type Kolmogorov inequality.

**Theorem.** Let $f \in C$ and

$$\lim_{y \to \infty} \frac{\log|f(iy)|}{y} = \lim_{y \to \infty} \frac{\log|f(-iy)|}{y} = a.$$ 

Then

$$\lim_{r \to \infty} \frac{N_+(r)}{r} = \lim_{r \to \infty} \frac{N_-(r)}{r} = \frac{a}{\pi}.$$ 

The Levinson theorem establishes the asymptotic density of the set of zeros of an entire function $f(z)$ of exponential type, satisfying certain boundedness conditions on the real line. Here we consider not the most general form of the theorem, but the spectral case where it is assumed that $f$ has the property

$$\int_{-\infty}^{\infty} \frac{\log|f(x)|}{1 + x^2} \, dx < \infty \quad (1)$$

(1) holds in many applications, so that the result given below is useful. Cf. [1, Chap. 5] or [2, Chap. 8] for the most general form of the theorem. The proposition with which we are concerned here was first proved by Miss Cartwright.

One says that $f(z)$ is of exponential type if

$$\log|f(z)| \leq O(|z|) \quad (2)$$

for large values of $|z|$. It suffices to consider the special situation where

$$\lim_{y \to \infty} \sup_{y \to \infty} \frac{\log|f(iy)|}{y} = \lim_{y \to \infty} \sup_{y \to \infty} \frac{\log|f(-iy)|}{y} = a. \quad (3)$$

(Here $y$ denotes a real variable.) One can always reduce the general case of a function $f$ which satisfies (1) and (2) to this by replacing $f(z)$ by $e^{icz}f(z)$ for a suitable choice of the real constant $c$.

We denote by $N_+(r)$ the number of zeros of $f(z)$ in modulus $\leq r$ with nonnegative real part, and let $N_-(r)$ be the number of these zeros in modulus $\leq r$ with negative real part. Here is the result with which we are concerned:

**Theorem.** Let $f$ be an entire function satisfying (1), (2), and (3). Then

$$\frac{N_+(r)}{r} \to \frac{a}{\pi} \quad \text{and} \quad \frac{N_-(r)}{r} \to \frac{a}{\pi} \text{ as } r \to \infty.$$ 

**1° Special Case.** All zeros of $f(z)$ are real.

Here $\log|f(z)|$ is harmonic in the half plane $\text{Im} \, z > 0$ and $\log f(z)$ can be defined there as an analytic function. Thus the function $\arg f(z) = \text{Im} \log f(z)$ is defined for $\text{Im} \, z > 0$.

We set, for Im\(z\) > 0,
\[
U(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log^+ |f(t)| \, dt,
\]
(4)
where the integral is finite by (1). The function \(U(z)\) is positive and harmonic in the upper half plane, continuous up to the real line, since \(\log^+ |f(t)|\) is continuous. Hence, the function
\[
W(z) = \log |f(z)| - U(z)
\]
is harmonic in \(\{\Im z > 0\}\), has boundary values \(\leq 0\) everywhere on \(\mathbb{R}\). By (2), \(W(z) < C(|z| + 1)\) for \(\Im z > 0\), and further, by (3),
\[
\lim_{y \to \infty} \sup \frac{W(iy)}{y} = a.
\]
(6)
Whence, by the familiar Phragmen-Lindelöf theorem, [2, Sec. 6.2, p. 82], one can conclude that
\[
W(z) < a \Im z, \quad \Im z > 0.
\]
(7)
From (7) and (6) we find the Poisson representation:
\[
W(z) = a \Im z + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} W(t) \, dt, \quad \Im z > 0,
\]
in view of the fact that the function \(\log |f(t)|\) is continuous on \(\mathbb{R}\) except for isolated logarithmic singularities. Returning to (5) and (4), we get
\[
\log |f(z)| = a \Im z + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log |f(t)| \, dt, \quad \Im z > 0.
\]
(8)
Derivation of the relation analogous to (8) can be made in the half plane \(\Im z < 0\). Thus, we see by (3) that \(\log |f(z)|\) is also given by the right side of (8) for \(\Im z > 0\). From this it follows that the function \(f(z)/f(\overline{z})\), being analytic in the half plane \(\Im z > 0\), has constant modulus there and hence is itself constant. Hence, the boundary value \(\arg f(x) = \lim_{y \to 0^+} \arg f(x + iy)\) is also constant on any segment of the real line not containing zeros of the function \(f(z)\). When \(x\) increases, passing through one of these zeros, \(\arg f(x)\) decreases by \(\pi\) times the multiplicity of this zero.

Now suppose for \(t \geq u\), \(n(t)\) is the number of zeros of \(f(z)\) on \([0, t]\) (counting their multiplicities), and for \(t < u\), \(n(t)\) is minus the number of these zeros on \([t, u]\) (counting their multiplicities). It is clear from what has just been said that
\[
\arg f(x) = -\lim_{y \to 0^+} \arg f(x + iy) = -\pi n(x) + \text{const}, \quad x \in \mathbb{R}.
\]
(9)
The function \(\arg f(z) = \Im \log f(z)\) is harmonic in the upper half plane; it is the harmonic conjugate of \(\log |f(z)|\). Since \(a \Re z\) is the harmonic conjugate of \(-a \Im z\), we have, by (8),
\[
\arg f(z) + a \Re z = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\Re z - t + \frac{t}{|t|^2 + 1}) \log |f(t)| \, dt + \text{const}, \quad \Im z > 0.
\]
In this relation we let \(y = \Im z\) tend to zero for a fixed value \(x = \Re z\), where \(f(x) \neq 0\). The limit value of the right side is found here by completely elementary means, because \(\log |f(t)|\) is an infinitely differentiable function in a neighborhood of the point \(x\). Letting
\[
\Delta (x) = n(x) - \frac{a}{\Re z} x,
\]
we find, in view of (9),
\[
\Delta (x) = -\frac{1}{\Re z} (\pi p) \int_{-\infty}^{\infty} (\frac{1}{x^2 - t^2} + \frac{t}{|t|^2 + 1}) \log |f(t)| \, dt + \text{const}
\]
(10)