One says that an entire function $f$ of finite exponential type belongs to the Cartwright class $C$, if
\[ \int_{-\infty}^{\infty} \frac{\log |f(x)|}{1 + x^2} \, dx < \infty. \]

Let $N_+(r)(N_-(r))$ denote the number of zeros of the function $f$ in the disk $|z| \leq R$, such that $\text{Re} \, z \geq 0$ ($\text{Re} \, z < 0$, respectively). We give a simple derivation of the following result of importance in the theory of entire functions, from a weak type Kolmogorov inequality.

**Theorem.** Let $f \in C$ and
\[ \lim_{y \to \infty} \frac{\log |f(iy)|}{y} = \lim_{y \to \infty} \frac{\log |f(-iy)|}{y} = a. \]

Then
\[ \lim_{r \to \infty} \frac{N_+(r)}{r} = \lim_{r \to \infty} \frac{N_-(r)}{r} = \frac{a}{\pi}. \]

The Levinson theorem establishes the asymptotic density of the set of zeros of an entire function $f(z)$ of exponential type, satisfying certain boundedness conditions on the real line. Here we consider not the most general form of the theorem, but the spectral case where it is assumed that $f$ has the property
\[ \int_{-\infty}^{\infty} \frac{\log |f(x)|}{1 + x^2} \, dx < \infty \tag{1} \]

(1) holds in many applications, so that the result given below is useful. Cf. [1, Chap. 5] or [2, Chap. 8] for the most general form of the theorem. The proposition with which we are concerned here was first proved by Miss Cartwright.

One says that $f(z)$ is of exponential type if
\[ \log |f(z)| < O(|z|) \tag{2} \]
for large values of $|z|$. It suffices to consider the special situation where
\[ \lim \sup_{y \to \infty} \frac{\log |f(iy)|}{y} = \lim \sup_{y \to \infty} \frac{\log |f(-iy)|}{y} = a. \tag{3} \]

(Here $y$ denotes a real variable.) One can always reduce the general case of a function $f$ which satisfies (1) and (2) to this by replacing $f(z)$ by $e^{icz}f(z)$ for a suitable choice of the real constant $c$.

We denote by $N_+(r)$ the number of zeros of $f(z)$ in modulus $\leq r$ with nonnegative real part, and let $N_-(r)$ be the number of these zeros in modulus $\leq r$ with negative real part. Here is the result with which we are concerned:

**Theorem.** Let $f$ be an entire function satisfying (1), (2), and (3). Then
\[ \frac{N_+(r)}{r} \to \frac{a}{\pi} \quad \text{and} \quad \frac{N_-(r)}{r} \to \frac{a}{\pi} \quad \text{as} \quad r \to \infty. \]

**Special Case.** All zeros of $f(z)$ are real.

Here $\log |f(z)|$ is harmonic in the half plane $\text{Im} \, z > 0$ and $\log f(z)$ can be defined there as an analytic function. Thus the function $\text{arg} \, f(z) = \text{Im} \, \log f(z)$ is defined for $\text{Im} \, z > 0$. 

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We set, for \( \text{Im} z > 0 \),
\[
U(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\text{Im} z}{|z-t|^2} \log^+ |f(t)| \, dt,
\]  
where the integral is finite by (1). The function \( U(z) \) is positive and harmonic in the upper half plane, continuous up to the real line, since \( \log^+ |f(t)| \) is continuous. Hence, the function
\[
W(z) = \log |f(z)| - U(z)
\]
is harmonic in \( \{ \text{Im} z > 0 \} \), has boundary values \( \leq 0 \) everywhere on \( \mathbb{R} \). By (2), \( W(z) < \mathcal{O}(|z| + 1) \) for \( \text{Im} z > 0 \), and further, by (3),
\[
\lim_{y \to -\infty} \sup \frac{W(iy)}{y} = -a.
\]
Whence, by the familiar Phragmen–Lindelöf theorem, [2, Sec. 6.2, p. 82], one can conclude that
\[
W(z) < a \text{Im} z, \quad \text{Im} z > 0.
\]
From (7) and (6) we find the Poisson representation:
\[
W(z) = a \text{Im} z + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\text{Im} z}{|z-t|^2} W(t) \, dt, \quad \text{Im} z > 0,
\]
in view of the fact that the function \( \log |f(t)| \) is continuous on \( \mathbb{R} \) except for isolated logarithmic singularities. Returning to (5) and (4), we get
\[
\log |f(z)| = a \text{Im} z + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\text{Im} z}{|z-t|^2} \log |f(t)| \, dt, \quad \text{Im} z > 0.
\]
Derivation of the relation analogous to (8) can be made in the half plane \( \text{Im} z < 0 \). Thus, we see by (3) that \( \log |f(z)| \) is also given by the right side of (8) for \( \text{Im} z > 0 \). From this it follows that the function \( f(z)/\overline{f(z)} \), being analytic in the half plane \( \text{Im} z > 0 \), has constant modulus there and hence is itself constant. Hence, the boundary value \( \arg f(x) = \lim_{y \to 0^+} \frac{\arg f(x+iy)}{y} \) is also constant on any segment of the real line not containing zeros of the function \( f(z) \). When \( x \) increases, passing through one of these zeros, \( \arg f(x) \) decreases by \( \pi \) times the multiplicity of this zero.

Now suppose for \( t \geq u \), \( n(t) \) is the number of zeros of \( f(z) \) on \([0, t]\) (counting their multiplicities), and for \( t < u \), \( n(t) \) is minus the number of these zeros on \([t, u]\) (counting their multiplicities). It is clear from what has just been said that
\[
\arg f(x) = \lim_{y \to 0^+} \frac{\arg f(x+iy)}{y} = -\pi n(x) + \text{const}, \quad x \in \mathbb{R}.
\]
The function \( \arg f(z) = \text{Im} \log f(z) \) is harmonic in the upper half plane; it is the harmonic conjugate of \( \log |f(z)| \). Since \( a \text{Re} z \) is the harmonic conjugate of \( -a \text{Im} z \), we have, by (8),
\[
\arg f(z) + a \text{Re} z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\text{Re} z - t}{|z-t|^2} + \frac{t}{t^2 + 1} \right) \log |f(t)| \, dt + \text{const}, \quad \text{Im} z > 0.
\]
In this relation we let \( y = \text{Im} z \) tend to zero for a fixed value \( x = \text{Re} z \), where \( f(x) \neq 0 \). The limit value of the right side is found here by completely elementary means, because \( \log |f(t)| \) is an infinitely differentiable function in a neighborhood of the point \( x \). Letting
\[
\Delta(x) = n(x) - \frac{a}{\sqrt{x}} x,
\]
we find, in view of (9),
\[
\Delta(x) = -\frac{1}{2\sqrt{x}} (\nu \rho) \int_{-\infty}^{\infty} \left( \frac{1}{x^2 - t^2} + \frac{t}{t^2 + 1} \right) \log |f(t)| \, dt + \text{const}
\]