OPERATOR APPROACH TO WEIGHTED ESTIMATES OF SINGULAR INTEGRALS

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In this paper we give a geometric approach (using only the theory of operators in Hilbert space) to $L^2$-weighted estimates of singular integral operators. In this way we are able to get an abstract operator theorem, a special case of which is the familiar theorem of Koosis, and also a generalization of Koosis' theorem to the case of operator-valued weights.

INTRODUCTION

The problem of weighted estimates of the Hilbert transform $H$, the Riesz projector $P_+$ or the Dirichlet kernels $\{f_{k,n}\}$ has recently attracted the attention of specialists. The classical formulation is for a given (finite, Borel) measure $\mu$ on the unit circle $T$, $T \equiv \{ \xi \in C : |\xi| = 1 \}$ to describe (completely or partially) the sets $H(\mu)$, $R(\mu)$, $D(\mu)$ of those measures $\nu$, for which respectively:

- $a) \int_T |f|^2 d\nu \leq C \int_T |f|^2 d\mu$, $\forall f \in \text{Pol}$
- $b) \int_T |P_+ f|^2 d\nu \leq C \int_T |f|^2 d\mu$, $\forall f \in \text{Pol}$
- $c) \int_T |P_{k,n} f|^2 d\nu \leq C \int_T |f|^2 d\mu$, $\forall f \in \text{Pol}$ $\forall k, n, k, n \in \mathbb{Z}$.

Here $\text{Pol}$ is the set of all trigonometric polynomials

$$f = \sum \hat{f}(n) z^n, \quad (z(\xi) = \xi, \quad \xi \in T)$$

$$Hf \equiv i \sum_{k<0} \hat{f}(k) z^{-k} - i \sum_{k>0} \hat{f}(k) z^k$$

$$P_+ f \equiv \sum_{k \geq 0} \hat{f}(k) z^{-k}, \quad P_{k,n} f \equiv \sum_{j=k}^{n} \hat{f}(j) z^j.$$

It is clear that in this formulation the requirements $a)-c)$ are equivalent, since

$$H = \frac{i}{2} (I - 2P_+), \quad P_{k,n} = S_k P_+ S^{-k} - S^{n+1} P_+ S^{-n-1},$$

where $Sf = z \cdot f$, $S$ is a unitary operator in $L^2(\mu)$ and $L^2(\nu)$. Consequently, $H(\mu) = R(\mu) = D(\mu)$. It is well known (cf., e.g., [5]) that the measure $\nu, \nu \in H(\mu)$ must be absolutely continuous with respect to the Lebesgue measure and $H(\mu) = H(\mu_\alpha)$; here $\mu_\alpha$ is the absolutely continuous part of the measure $\mu$. Hence, without loss of generality one can assume both measures $\mu$ and $\nu$ are absolutely continuous.

Now we formulate the basic known results on weighted estimates. Here and below we shall $d\mu = \omega dm$, $d\nu = \nu dm$, $\omega, \nu \in L^1(T), \nu > 0$, $m$ is the normalized Lebesgue measure on $T$. 

THEOREM I (Helson--Szegö).
\[ \mu \in H(\mu) \iff \mathcal{W} = \exp (u_1 + H u_2), \]
where \( u_1, u_2 \) are bounded real functions and \( \|u_2\|_\infty < \pi/2. \)

THEOREM II (Hunt--Muckenhoupt--Wheeden).
\[ \mu \in H(\mu) \iff \sup_I \left( \frac{1}{m(I)} \int_I \mathcal{W} \right) < \infty \]
(Here the supremum is taken over all arcs \( I \) of the circle \( \mathbb{T} \).)

THEOREM III (Koosis).
\[ H(\mu) \neq \{0\} \iff \frac{1}{\mathcal{W}} \in L^4(\mathbb{T}) \]

THEOREM IV (Cotlar--Arocena--Sadoski).
\[ \forall \mu \in H(\mu) \iff \exists M > 0, \exists h \in H^4 \text{ such that:} \]
\[ |M \mathcal{W} + h - \mathcal{W}| \leq M \mathcal{W} - \mathcal{W}. \]

Here \( H^4 \) is the Hardy space,
\[ H^4 \text{ def } \{ f \in L^4(\mathbb{T}) : \hat{f}(n) = 0, n < 0 \}. \]

Theorem I was first proved in [1] (cf. also [9, p. 254]) by the method of the theory of Hardy classes. The proof of Theorem II (cf. [2, 3]) is based on the use of refined estimates of singular integrals.

Several proofs of Theorem III are known: Koosis' own proof ([4]), uses the methods of the theory of analytic functions; the proof of Carleson and Jones [7] is similar in idea to the proof of Theorem II; and the proof of Cotlar-Arocena-Sadoski, is based on a lifting theorem due to the same authors. Theorem IV is proved in this same spirit (cf. [5]). In general one can speak of Cotlar's method, which is of interest in its own right, and turns out to be useful in many questions.

In the various proofs of Theorem III one also shows that if \( H(\mu) \neq \{0\} \), then there exist in \( H(\mu) \) "sufficiently larger" measures: Koosis ([4]) showed that in this case the measure \( \nu \in H(\mu) \) can be so chosen that \( \log \nu \in L^4(\mathbb{T}) \); Carleson and Jones ([7]) showed that if \( \frac{\hat{\mu}}{\mu} \in L^p(\mathbb{T}), 1 < p < \infty \) then for any \( \delta > 0 \) there exists a density \( \nu \), for which \( \nu \, \mathcal{W} \, dm \in H(\mu) \) and \( \frac{1}{\nu} \in L^{p-\delta} \).

In the present paper we give a purely geometric approach (using only the theory of operators in Hilbert space) to estimates of type a)-c), allowing us to get a certain abstract version of Koosis' theorem (Theorem III), Theorem 1, and also a generalization of Koosis' result to the case of operator-valued weights \( V \) and \( W \) (Theorem 2). We also prove another theorem (Theorem 3), which is a "noncommutative analog" of part of the result of Carleson and Jones (\( p = 2, \delta = 1 \)).

Some of the results of this paper were announced in [10].

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1. Formulation of the Problem

We note first of all that the estimates a)-c) can be rewritten in the form
\[ a') \|R_\alpha f \| \leq C \cdot |R_1 f|, \forall f \in Pol \]