A REFLEXIVITY CRITERION FOR STALKS OF A BANACH BUNDLE

A. V. Koptev

In the article, we deal with continuous Banach bundles (CBB) over extremely disconnected compact spaces (EDCS). The interest in such objects is generated by a strong possibility of developing the theory of lattice-normed spaces (LNS) by studying CBBs. For functional realization of LNSs, A. E. Gutman introduced the notion of an ample (or complete) Banach bundle (see [1]) which is a CBB over an EDCS such that each of its continuous bounded sections over an everywhere dense set extends to a global continuous section. Moreover, for every CBB $\mathcal{X}$ over an EDCS $Q$ there is a (unique) ample span, namely, an ample bundle $\bar{\mathcal{X}}$ over $Q$ such that $\mathcal{X}$ is a subbundle of $\bar{\mathcal{X}}$ and every element $u$ of the space $C(Q, \bar{\mathcal{X}})$ of all global continuous sections of the bundle $\bar{\mathcal{X}}$ takes the values $u(q) \in \mathcal{X}(q)$ in a comeager set. To the dual LNS there corresponds the dual CBB: if $\mathcal{X}$ is an ample bundle over an EDCS $Q$ then a CBB $\mathcal{X}'$ is dual to $\mathcal{X}$ if $C(Q, \mathcal{X}') = \{u' : \forall q \in Q \ u'(q) \in \mathcal{X}(q)', \forall u \in C(Q, \mathcal{X}) \ (u(q)|u'(q)) \in C(Q)\}$. Each stalk $\mathcal{X}'(q)$ of the dual bundle $\mathcal{X}'$ is a subspace of the dual $\mathcal{X}(q)'$ of the stalk $\mathcal{X}(q)$. The natural question as to when the equality $\mathcal{X}'(q) = \mathcal{X}(q)'$ holds was resolved in [1] as follows: In the most cases, this is so if and only if the space $\mathcal{X}(q)$ is reflexive. The present article is further research into the question of the equality $\mathcal{X}'(q) = \mathcal{X}(q)'$.

Let $X$ be a Banach space and let $\mathcal{X}$ be the ample span of the constant bundle with stalk $X$. Originally, the author sought for the properties of $X$ that would guarantee reflexivity for the stalks of $\mathcal{X}$. It is well known from nonstandard analysis (see [2]) that the reflexivity of the nonstandard hull of a Banach space is equivalent to the superreflexivity of the initial space. At the same time, each stalk of $\mathcal{X}(q)$ is in some sense similar to the nonstandard hull of $X$. These circumstances prompt us to suppose that the reflexivity of the stalks of $\mathcal{X}(q)$ is equivalent to the superreflexivity of $X$. Theorem 15 corroborates the conjecture. Some other conditions equivalent to the reflexivity of the stalks $\mathcal{X}(q)$ are listed in Theorem 16. During the proof of the theorems, the following curious fact was revealed: each stalk $\mathcal{X}(q)$ is finitely representable in $X$ (Theorem 9). Consequently, in a sense, each stalk $\mathcal{X}(q)$ consists of the same finite-dimensional spaces as $X$. At the same time, a stalk $\mathcal{X}(q)$ is “close” to the nonstandard hull of $X$ which is essentially “larger” than $X$ in its categorical and topological characteristics. Moreover, in the majority of cases the reflexivity of a stalk of an arbitrary ample CBB is equivalent to the superreflexivity of the stalk. The idea behind the proof of this fact is borrowed from [2].

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1. DEFINITION. Consider Banach spaces $G$ and $Y$ and a number $\lambda \geq 1$. A linear operator $T : G \rightarrow Y$ is a $\lambda$-embedding if it possesses one of the following equivalent properties:

1. $\|g\| \leq \|Tg\| \leq \lambda \|g\|$ for all $g \in G$;
2. $1 \leq \|Tg\| \leq \lambda$ for all $g$ on the unit sphere of the space $G$;
3. the operator $T$ is invertible as an element of $L(G, \text{Im} T)$, $\|T\| \leq \lambda$, and $\|T^{-1}\| \leq 1$.

We say that a Banach space $X$ is finitely representable in $Y$ if, for every finite-dimensional normed subspace $G \subset X$ and every positive real $\varepsilon$, there exists a $(1 + \varepsilon)$-embedding of $G$ into $Y$.

2. We list some properties of finite representability.

(a) Every Banach space is finitely representable in itself.

(b) Banach subspaces of a Banach space are finitely representable in the space.

(c) If a Banach space $X$ is finitely representable in a Banach space $Y$ then every Banach subspace of $X$ is finitely representable in $Y$. 

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(d) If Banach spaces $X$ and $Y$ are isometric then they are finitely representable in one another.

(e) Let $X$, $Y$, and $Z$ be Banach spaces; moreover, $X$ is finitely representable in $Y$ and $Y$ is finitely representable in $Z$. Then $X$ is finitely representable in $Z$.

(f) A Banach space $X$ is finitely representable in a finite-dimensional Banach space $Y$ if and only if $X$ is isometrically embeddable in $Y$.

We have just to prove criterion (f), and only its necessity part. Let a Banach space $X$ be finitely representable in a finite-dimensional Banach space $Y$. Given a linear operator $T \in L(X, Y)$, introduce the continuous function $f_T$ on the unit sphere $S_X$ of the space $X$ as follows: $f_T(x) := \|Tx\| - 1$. By the definition of finite representability, we have $\dim X \leq \dim Y$. Therefore, $S_X$ is a compact set. Consider the function $h : L(X, Y) \to \mathbb{R}$ given as $h(T) := \|f_T\|_\infty$ for all $T \in L(X, Y)$. The function is obviously continuous. The space $L(X, Y)$ is finite-dimensional. Therefore, the set $\{T \in L(X, Y) : \|T\| \leq 2\}$ is compact and the continuous function $h$ attains its infimum on this set. Since $X$ is finitely representable in $Y$, the infimum equals zero; i.e., $h(T) = 0$ for some operator $T \in L(X, Y)$ which is a sought isometry.

3. Definition. A Banach space is superreflexive if every Banach space finitely representable in it is reflexive. It is well known that every Hilbert space is superreflexive.

4. We list some properties of superreflexive spaces (see [2]).

(a) A superreflexive Banach space is reflexive.

(b) A Banach space finitely representable in a superreflexive Banach space is superreflexive itself.

(c) Banach subspaces of a superreflexive Banach space are superreflexive.

(d) For a Banach space $X$ to be superreflexive, it necessary and sufficient that every separable Banach space finitely representable in $X$ be reflexive.

5. Henceforth we use the following notation. Given a vector space $E$, a collection $e = (e_1, \ldots, e_n)$ of vectors in $E$, and an element $A = (A_1, \ldots, A_n)$ in $\mathbb{R}^n$, we put $A e := \sum_{k=1}^{n} \lambda_k e_k$.

6. Lemma. Assume that $n$ is a natural number; $\Lambda$, a bounded subset of $\mathbb{R}^n$; $Q$, a topological space; $\mathcal{X}$, a CBB over $Q$; and $u = (u_1, \ldots, u_n)$, a collection of sections in $C(Q, \mathcal{X})$. Then, for every real $\delta > 0$ and every point $q \in Q$, there is a neighborhood $U$ of $q$ in which $\|\lambda u\| - \|\lambda u(q)\| < \delta$ for every $\lambda \in \Lambda$.

Proof. Without loss of generality we can proceed with the norm $\|\cdot\|_1$ in $\mathbb{R}^n$ (the sum of the moduli of components). Take a finite $\varepsilon$-net $\Lambda_0$, $\Lambda_0 \subset \Lambda$, where $\varepsilon := \delta/(2\max\{2\|u_k(q)\| + 1 : k = 1, \ldots, n\})$. Find a neighborhood $U$ of $q$ in which $\|u_k\| \leq \|u_k(q)\| + 1$ for all $k = 1, \ldots, n$ and $\|\lambda_0 u\| - \|\lambda_0 u(q)\| \leq \delta/2$ for all $\lambda_0 \in \Lambda_0$. Let $\lambda$ be an arbitrary element of $\Lambda$. Choose $\lambda_0 \in \Lambda_0$ so as to have $\|\lambda - \lambda_0\|_1 < \varepsilon$. Then the following relations hold for all $p \in U$:

$$\|\lambda u\|(p) - \|\lambda u(q)\| \leq \|\lambda u\|(p) - \|\lambda_0 u\|(p) + \|\lambda_0 u\| - \|\lambda_0 u(q)\|$$

$$+ \|\lambda_0 u\| - \lambda u\| \leq \|\lambda u\| - \lambda_0 u\|(p) + \frac{\delta}{2} + \|\lambda_0 u - \lambda u\| \leq \frac{\delta}{2}$$

$$+ \sum_{k=1}^{n} |\lambda_k - \lambda_0k| \|u_k\| + \|u_k\|(q)) \leq \frac{\delta}{2} + \|\lambda - \lambda_0\|_1 \max\{2\|u_k(q)\| + 1\} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus, $U$ is a sought neighborhood.

7. Lemma. Let $\mathcal{X}$ be a CBB over a topological space $Q$. Fix a point $q \in Q$ and a finite-dimensional subspace $G \subset \mathcal{X}(q)$. Let a collection $u = (u_1, \ldots, u_n)$ of global continuous sections of $\mathcal{X}$ be such that $u(q) := (u_1(q), \ldots, u_n(q))$ is a basis for $G$ on the unit sphere. Put $\Lambda := \{\lambda \in \mathbb{R}^n : \|\lambda u(q)\| = 1\}$. Then

1. for every $\delta > 0$, there is a neighborhood $U$ of $q$ in which $1 - \delta \leq \|\lambda u\| \leq 1 + \delta$ for all $\lambda \in \Lambda$;
2. for every $\varepsilon > 0$, there exists an operator $T \in L(G, C(Q, \mathcal{X}))$ and a neighborhood $U$ of $q$ such that $\text{Im} T = \text{lin} u$ and $1 \leq \|Tg\| \leq 1 + \varepsilon$ on $U$ for every vector $g$ on the unit sphere of $G$.

Proof. (1): It is obvious that $\Lambda$ is the unit sphere for some norm in $\mathbb{R}^n$. By Lemma 6, there is a neighborhood $U$ of $q$ in which $\|\lambda u - 1\| = \|\lambda u - \|\lambda u\|(q)\| < \delta$ for all $\lambda$ in $\Lambda$.

2: Given an $\varepsilon$, find a $\delta > 0$ satisfying the condition $(1 + \delta)/(1 - \delta) \leq 1 + \varepsilon$. Given the $\delta$, find a neighborhood $U$ fitting item (1). We have $1 \leq \|\lambda u\|/(1 - \delta) \leq (1 + \delta)/(1 - \delta) \leq 1 + \varepsilon$ in $U$. Define $T$ as