This paper is a direct continuation of the author's previous paper. One considers questions concerning the uniqueness and the solvability of equations of the higher approximations of the WKB method and the relation between these questions and the properties of the averaged Lagrangian.

4. The present paper is a direct continuation of [2]. Their continuity will apply to the general system of notations and references as well as to the numbering of the sections, lemmas, and formulas. Thus, for example, all the formulas of the form \((i,j)\), where \(i = 0, 1, 2, 3, j \geq 1\), refer to the first part of our work. It should be noted that, just as in the first part, the author has proceeded from the methods and results of Maslov and Dobrokhotov's paper [1].

Before proceeding to the subsequent presentation, we make some remarks. Firstly, everywhere in the second part we shall consider the single-phase case \(I = 1\), leading to an ordinary differential equation with respect to \(\tau\).

Secondly, very often, without any special mention, we shall switch from the Lagrangian notation of the equation (1.4) to the Hamiltonian notation. We elucidate this passage in a more detailed manner. Introducing the variables

\[ P(\tau, x, \epsilon) = \varphi^i (U, U_{\tau}, \epsilon v_x U, x) = \varphi^i \epsilon v_x U, \]

we can switch to the Hamiltonian notation of the variational equation (1.4):

\[ \delta_{0, \infty} \int_0^{2\pi} d\tau \{ \mathcal{F} \mathcal{U} \mathcal{F} - \mathcal{H} (U, P, \epsilon v_x U, x) \} = 0. \]  

Moreover, by virtue of the fact that \(U(\tau, x, \epsilon)\) represents the series (1.6) with respect to the powers of \(\epsilon\), the function \(P(\tau, x, \epsilon)\) also represents a series

\[ \sum_{j \geq 0} \epsilon^j p_j (\tau, x), \]

and all \(p_j(\tau, x)\) are \(2\pi\)-periodic with respect to \(\tau\).

The variational equations (1.12), which, strictly speaking, is liable to discussion, takes the form

\[ \delta_{0, \infty} \int_0^{2\pi} d\tau \{ \mathcal{F} \mathcal{U} \mathcal{F} - \mathcal{H} (P, \epsilon v_x U, x) \} = O(\epsilon^{N+1}). \]  

Expanding the integrand from the left-hand side of (4.4) into a series of powers of \(\epsilon\), we obtain that, in order that equality (4.4) be satisfied, it is necessary and sufficient that

\[ \delta_{0, \infty} \int_0^{2\pi} d\tau \{ \mathcal{F} \mathcal{U} \mathcal{F} - H_j (y_0, y_1, \ldots, y_N, \epsilon v_x U, x) \} = 0, \]

for \(j = 0, 1, \ldots, N\).
Here $H_0$ is the term of order $\varepsilon$ in the expansion of $H(U, P, \theta, x)$ in a series of powers of $\varepsilon$ (see a similar expansion in the first part of the paper, namely in formula (1.11)). We note that making use of the same considerations which have led to the equations (1.14), one can show that from the $2N(N + 1)$ Euler equations for system (4.5), $2(N + 1)$ equations are linearly independent, obtained by taking the variation of the initial integral with respect to $y_0, p_0$:

$$
\delta_{y_0, p_0} \left\{ \frac{d}{d\tau} \left[ \sum_{i=0}^{N-1} \frac{\partial}{\partial y_i} \left( \frac{\partial H}{\partial p_i} \right) - \frac{\partial H}{\partial y_i} \right] \right\} = 0,
$$

where $s = 0, 1, \ldots, N$.

Moreover, in the first part of our paper we have investigated the question of the solvability of the equations of zero approximation (see Lemma 1). As far as the subsequent approximations are concerned, when working in the Hamiltonian form the corresponding equations take the form:

$$
\mathcal{J} \frac{d}{d\tau} \left( \begin{array}{c} y_0 \\ \vdots \\ y_{N-1} \\ p_0 \\ \vdots \\ p_{N-1} \end{array} \right) + \mathcal{Z} \left( \begin{array}{c} y_0 \\ \vdots \\ y_{N-1} \\ p_0 \\ \vdots \\ p_{N-1} \end{array} \right) = \left( \begin{array}{c} \mathcal{F}_{y_0} \\ \vdots \\ \mathcal{F}_{y_{N-1}} \\ \mathcal{F}_{p_0} \\ \vdots \\ \mathcal{F}_{p_{N-1}} \end{array} \right),
$$

(4.7)

where $\mathcal{J} = \begin{pmatrix} 0 & 0 \\ -E & 0 \end{pmatrix}$, $E$ is the $n \times n$ identity matrix and the matrix $\mathcal{Z}$ is equal to \begin{pmatrix} \mathcal{H}^{y_0} & \mathcal{H}^{p_0} \\ \mathcal{H}^{y_{N-1}} & \mathcal{H}^{p_{N-1}} \end{pmatrix}.

$\mathcal{F}_{y_0}$ and $\mathcal{F}_{p_0}$ represent $n$-dimensional vector-functions depending on $y_0, \ldots, y_{N-1}, p_0, \ldots, p_{N-1}$. In the Lagrangian form, the analogues of Eqs. (4.7) are Eqs. (1.8).

We shall assume that the equations of zero approximation are completely integrable. Then, switching to the variables action-angle, we rewrite Eq. (4.6) in the following manner:

$$
\delta_{y_0, p_0} \left\{ \frac{d}{d\tau} \left[ \sum_{i=0}^{N-1} \frac{\partial}{\partial y_i} \left( \frac{\partial H}{\partial p_i} \right) - \frac{\partial H}{\partial y_i} \right] \right\} = 0,
$$

(4.6)

where $s = 0, 1, \ldots, N$.

In the variables action-angle we have $H_0(y_0, y_{N-1}, x) = H_0(y_0, y_{N-1}, x)$, so that in the matrix $\mathcal{Z}$ only the upper left block, corresponding to $H_0$, is different from zero and by virtue of the constancy of $y_0$ along the trajectory, $H_0$ is a matrix that depends only on $x$ and is constant with respect to $\tau$. Finally, in order to proceed further, we make assumptions regarding the nondegeneracy for all $x$ (at least locally) of the matrix $H_0(\mathcal{J}, \theta, x)$.

5. We pose the problem of the solvability of Eqs. (4.7) or, in Lagrangian form, (1.8). By virtue of the self-adjointness in $L_2(0, 2\pi)$ of the operators $\mathcal{J} \frac{d}{d\tau} + \mathcal{Z}$ (or of the operator $M$ of form (1.10)), for the solvability of these equations it is necessary and sufficient that the right-hand sides $((\mathcal{F}_{y_0}, \mathcal{F}_{p_0})$ and the corresponding $f$ be orthogonal to the eigenspace of the operator $\mathcal{J} \frac{d}{d\tau} + \mathcal{Z}$ (resp. of the operator $M$). The eigenspace of the operator $\mathcal{J} \frac{d}{d\tau} + \mathcal{Z}$ is $n$-dimensional and the eigenfunctions are the vector-functions $\varphi_i: \varphi_i = \varphi_i, \quad \varphi_i = 0$; $i = 1, \ldots, n$.

To the same eigenfunctions $z_i(y_0, \mathcal{J}, \theta, x)$ one can arrive also in another way. We note that the solution of the nonlinear equation of zero approximation depends on $n$ arbitrary constants, whose roles are played by the initial phases

$$
y_0 = y_0(\mathcal{J}, \omega t + \varphi_{\text{Init}}, y_{N-1}, \theta, x).
$$

The derivatives of the obtained solutions with respect to these parameters give us $n$ linearly independent eigenfunctions of the operator $M$ (or, in the Hamiltonian form, of the operator $\mathcal{J} \frac{d}{d\tau} + \mathcal{Z}$), namely $\frac{\partial y_0}{\partial \varphi_{\text{Init}}} (\mathcal{J}, \omega t + \varphi_{\text{Init}}, y_{N-1}, \theta, x)$ or in the Hamiltonian form the functions $z_i(y_0, \mathcal{J}, \theta, x)$, which have been already written down. Therefore, the integrability conditions of Eqs. (4.7), (1.8) have the form