Existence, Uniqueness and Convergence as Vanishing Viscosity for a Reaction-Diffusion-Convection System

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Abstract. A simple qualitative model of dynamic combustion

\[
\begin{cases}
\frac{\partial (u+qz)}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial z}{\partial t} = -F(u, z)
\end{cases}
\]

is considered, where constant \( q > 0 \) represents the binding energy. The existence and uniqueness of classical solution for the initial value problem and for the first boundary value problem of system (1) are proved and then it is proved that as \( \varepsilon \to 0 \) classical solutions of the initial value problem of system (1) converge to a pair of limit functions, which is an admissible solution of the initial value problem of system (1) with \( \varepsilon = 0 \).

I. Introduction

A simple reaction-diffusion-convection system, introduced by Majda\textsuperscript{[5]} as a qualitative model of dynamic combustion, may be written in the form

\[
\begin{cases}
\frac{\partial (u+qz)}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial z}{\partial t} = -F(u, z), \\
(u, z)|_{t=0} = (u_0(x), z_0(x)), \\
(x, t) \in \mathbb{R}^T = \mathbb{R} \times (0, T),
\end{cases}
\]

where constants \( q > 0, \varepsilon > 0 \) represent the binding energy and viscosity, \( u \) is a lumped variable representing density, velocity and temperature, \( z \) is the fraction of unburnt gas, \( z_0(x) \geq 0 \), \( f(u) \) and \( F(u, z) \) are twice differentiable function satisfying

\[
\sup_{0 \leq z \leq N, \ |u| < +\infty} \frac{\partial F(u, z)}{\partial u} \leq K(N), \ F(u, 0) = 0 \ (|u| < \infty), \\
\frac{\partial F(u, z)}{\partial z} \geq 0 \ (|u| < \infty, z \geq 0),
\]

here \( K(N) \) being a constant depending on \( N \).
If $\varepsilon = 0$ the system (1.1), (1.2) reduces to a hyperbolic reaction-convection system

$$
\begin{aligned}
\frac{\partial (u + qz)}{\partial t} + \frac{\partial f(u)}{\partial x} &= 0, \\
\frac{\partial z}{\partial t} &= -F(u, z),
\end{aligned}
$$

$$
(u, z) |_{t=0} = (u_0(x), z_0(x)), \quad x \in \mathbb{R}.
$$

In this paper we will prove the existence and uniqueness of classical solution for the initial value problem and for the first boundary value problem of system (1.1), and then prove that as $\varepsilon \to 0$ the classical solution of (1.1), (1.2) converges to a pair of limit functions, which is an admissible solution of the initial value problem of system (1.4), (1.5). The main results in this paper are Theorem 2.1—2.2 and Theorem 4.2—4.3.

II. Existence and Uniqueness for Cauchy Problem and for First Boundary Problem of System (1.1)

We now define some Hölder spaces. The following notations will be adopted:

$$
\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_m = (-m, m), \quad \mathbb{R}^T = \mathbb{R} \times (0, T), \quad \mathbb{R}_m^T = \mathbb{R}_m \times (0, T).
$$

$$
C^{(k+\alpha, \frac{k+\alpha}{2})} (\mathbb{R}_m^T), \quad 0 < \alpha \leq 1, \quad k = 0, 1, 2, \quad \text{is the Banach space of functions } u(x, t) \text{ that are defined in } \mathbb{R}_m^T, \quad \text{and have a finite norm}
$$

$$
\| u \|_{C^{(k+\alpha, \frac{k+\alpha}{2})} (\mathbb{R}_m^T)} = \sum_{i=0}^{k} \| \partial_x^i u \|_{\mathbb{R}_m^T} + \sum_{i=0}^{k} \| [\partial_x^i u]^{(\alpha, \frac{k}{2})} \|_{\mathbb{R}_m^T}, \quad k = 0, 1,
$$

$$
\| u \|_{C^{(1+\alpha, \frac{1}{2})} (\mathbb{R}_m^T)} = \sum_{i=0}^{1} \| \partial_x^i u \|_{\mathbb{R}_m^T} + \sum_{i=0}^{1} \| [\partial_x^i u]^{(\alpha, \frac{1}{2})} \|_{\mathbb{R}_m^T} + \| \partial_t u \|_{\mathbb{R}_m^T} + \| [\partial_t u]^{(\alpha, \frac{1}{2})} \|_{\mathbb{R}_m^T},
$$

where

$$
\| \partial_x^i u \|_{\mathbb{R}_m^T} = \sup_{(x, t) \in \mathbb{R}_m^T} |\partial_x^i u(x, t)|,
$$

$$
[\partial_x^i u]^{(\alpha, \frac{k}{2})} = \sup_{(x, t) \in \mathbb{R}_m^T} \left( \frac{|\partial_x^i u(x, t) - \partial_x^i u(y, t)|}{|x - y| + |t - \tau|^{1/2}} \right)^{\alpha}.
$$

$C^{(1+\alpha, \frac{1}{2})} (\mathbb{R}_m^T)$ denotes the Hölder space with finite norm

$$
\| u \|_{C^{(1+\alpha, \frac{1}{2})} (\mathbb{R}_m^T)} = \| u \|_{\mathbb{R}_m^T}^{(0+1, \frac{0+1}{2})} + \| \partial_u u \|_{\mathbb{R}_m^T}^{(0+1, \frac{0+1}{2})}.
$$

$C^{(k+\alpha, \frac{k+\alpha}{2})} (\mathbb{R}_m^T)$ and $C^{(1+\alpha, \frac{1}{2})} (\mathbb{R}_m^T)$ are defined in a similar manner.