Finite Presentability of Steinberg Groups over Group Rings*

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Abstract. Let $A$ be a finitely generated commutative $\mathbb{Z}$-algebra with Krull dimension $d$, and let $\pi$ be an arbitrary finite group. It is proved that the Steinberg group $St_n(A\pi)$ is finitely presented whenever $n \geq 4$. If, in addition, $n \geq d+3$, and $K_1(A\pi)$ and $K_2(A\pi)$ are finitely generated, then $E_n(A\pi)$ and $GL_n(A\pi)$ are finitely presented.

Let $A$ be an associative ring with $1$, and $n$ be an integer $\geq 3$. The Steinberg group $St_n(A)$ is the group defined by generators $X_{ij}(a)(1 \leq i, j \leq n, i \neq j, a \in A)$ subject to the relations

$$X_{ij}(a)X_{ij}(b) = X_{ij}(a+b),$$

$$[X_{ij}(a), X_{il}(b)] = X_{ii}(ab), \quad i \neq l,$$

$$[X_{ij}(a), X_{kl}(b)] = 1, \quad j \neq k \text{ and } i \neq l.$$ 

This is, of course, an abstract model of the elementary group $E_n(A)$. We are concerned with the finite presentability of $St_n(A)$, $E_n(A)$ and $GL_n(A)$, i.e., we want to find conditions on $A$ and $n$ under which these groups are finitely presented.

First of all, if $A$ is commutative, then a necessary condition of finite presentability for $St_n(A)$ is that $A$ is finitely generated as a $\mathbb{Z}$-algebra, and hence, $A$ is a noetherian ring with finite Krull dimension $d$. Under this condition Rehmann and Soulé [5] proved that $St_n(A)$ is finitely presented for each $n \geq 4$. If furthermore $n \geq d+3$, and $K_1(A)$ and $K_2(A)$ are finitely generated, then $E_n(A)$ and $GL_n(A)$ are finitely presented. In this paper we prove a similar result in the case of group rings $A\pi$.

Main Theorem. Let $A$ be a finitely generated commutative $\mathbb{Z}$-algebra with Krull dimension $d$, and let $\pi$ be an arbitrary finite group. Then $St_n(A\pi)$ is finitely presented whenever $n \geq 4$. If, in addition, $n \geq d+3$, and $K_1(A\pi)$ and $K_2(A\pi)$ are finitely generated, then $E_n(A\pi)$ and $GL_n(A\pi)$ are finitely presented.

As the first step we consider a special case.

Proposition 1. Suppose that $A = \mathbb{Z}[T_1, \ldots, T_r]$ where $T_1, \ldots, T_r$ are independent indeterminates over $\mathbb{Z}$, and $\pi$ is a finite group. Then $St_n(A\pi)$ is finitely presented for each $n \geq 4$.

Proof. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ and $T = (T_1, \ldots, T_r)$ we write $T^\alpha$ =

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For a given positive integer $m$, let $G_m$ be the group generated by the symbols $X_{ij}(\lambda T^* \sigma)(1 \leq i, j \leq n, i \neq j, \lambda = \pm 1, |\lambda| \leq m, \sigma \in \pi)$ subject to the relations

\begin{align}
(1)_m & \quad X_{ij}(T^* \sigma)X_{ij}(-T^* \sigma) = 1, \\
(2)_m & \quad [X_{ij}(\lambda T^* \sigma), X_{jl}(\mu T^\gamma \tau)] = X_{il}(\lambda \mu T^{*+\beta} \sigma \tau), \quad i \neq l, |\alpha + \beta| \leq m, \\
(3)_m & \quad [X_{ij}(\lambda T^* \sigma), X_{kl}(\mu T^\gamma \tau)] = 1, \quad j \neq k \quad \text{and} \quad i \neq l.
\end{align}

Obviously, there exists a canonical group homomorphism $\varphi_m : G_m \rightarrow G_{m+1}$. Moreover, $\varphi_m$ is surjective by means of $(2)_{m+1}$. It is easily seen that every $G_m$ is finitely presented, and $\lim_m G_m = St_n(A\pi)$. Therefore, it suffices to show that $\varphi_m$ is in fact an isomorphism for each $m \geq 2$.

**Lemma 1.** Assume that

(i) $\alpha, \beta, \alpha', \beta' \in \mathbb{N}^r$ such that $|\alpha|, |\beta|, |\alpha'|, |\beta'| \leq m$, \( \alpha + \beta = \alpha' + \beta' \), and \( |\alpha + \beta| = m + 1 \),

(ii) $\sigma, \tau, \sigma', \tau' \in \pi$ with $\sigma \tau = \sigma' \tau'$, and

(iii) $\lambda, \mu, \lambda', \mu' = \pm 1$ with $\lambda \mu = \lambda' \mu'$.

Then, for $i \neq k$,

$$[X_{ij}(\lambda T^* \sigma), X_{jk}(\mu T^\gamma \tau)] = [X_{ij}(\lambda' T^* \sigma'), X_{jk}(\mu' T^\gamma \tau')]$$

holds in $G_m$.

**Proof.** Take $e = e(\alpha + \beta)$. Then $\alpha - e, \beta - e \in \mathbb{N}^r$, and $|\alpha + \beta - e| = m$ since $|\alpha|, |\beta| \leq m$ and $|\alpha + \beta| = m + 1$.

If $s = j$, we may choose $l \neq i, j, k$ since $n \geq 4$. Then

$$[X_{ij}(\lambda T^* \sigma), X_{lk}(\mu T^\gamma \tau)] = [X_{ij}(\lambda T^* \sigma), [X_{jl}(\mu T^{\gamma - \epsilon} \tau), X_{lk}(T^\epsilon)]]$$

$$= [X_{ij}(\lambda T^* \sigma), [X_{jl}(\mu T^{\gamma - \epsilon} \tau), X_{jk}(\mu T^\gamma \tau)X_{lk}(T^\epsilon)]]$$

$$= [X_{ij}(\lambda \mu T^{*+\beta-\epsilon} \sigma \tau), X_{lk}(T^\epsilon)X_{jk}(\mu T^\gamma \tau)]$$

$$= [X_{ij}(\lambda \mu T^{*+\beta-\epsilon} \sigma \tau), X_{lk}(T^\epsilon)].$$

Similarly,

$$[X_{ij}(\lambda' T^* \sigma'), X_{jk}(\mu' T^\gamma \tau')],$$

$$= [X_{ij}(\lambda' \mu' T^{*+\beta-\epsilon} \sigma' \tau'), X_{lk}(T^\epsilon)]$$

$$= [X_{ij}(\lambda \mu T^{*+\beta-\epsilon} \sigma \tau), X_{lk}(T^\epsilon)].$$

If $s \neq j$, replacing $l$ by $s$ in the above procedure, it is reduced to the above situation.

Let us continue the proof of Proposition 1. For $i \neq j$ and $\alpha \in \mathbb{N}^r$ with $|\alpha| = m + 1$, take $k \neq i, j$. Define

$$X_{ij}(\lambda T^* \sigma) = [X_{ik}(\lambda T^{* - \epsilon} \sigma), X_{kj}(T^\epsilon)]$$

in $G_m$, where $\epsilon = \epsilon(\alpha)$. By Lemma 1, the definition is independent of the choice of