On Incomplete Gaussian Sums

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Abstract. In this paper it is proved that
\[ \sum_{n=N+1}^{N+H} \chi(n) \psi(n) \ll H^{1-\frac{1}{r}} q^{\frac{1}{4}(r-1)+\varepsilon}, \]
where \( r = 4 \), \( q \) is a prime power, \( \chi \) and \( \psi \) are multiplicative and additive characters modulo \( q \) respectively, with \( \chi \) nontrivial.

Keywords. Multiplicative characters

1 Introduction

Let \( \chi \) and \( \psi \) be multiplicative and additive characters modulo \( q \) respectively. Let \( r \) be an integer greater than 2. Let \( \chi \) be nontrivial.

In this paper we prove

Theorem 1. If \( r = 4 \) and \( q \) is a prime power, then
\[ \sum_{n=N+1}^{N+H} \chi(n) \psi(n) \ll H^{1-\frac{1}{r}} q^{\frac{1}{4}(r-1)+\varepsilon}. \] (1)

The following results are due to Burgess\,[1,2].
A. (1) holds when \( r = 3 \);
B. (1) holds when \( r = 4 \) and \( q = p^\alpha \) is a prime power with \( \alpha \leq 7 \).

Let
\[ f_1(x) = \prod_{i=1}^{r} (x + m_i), f_2(x) = \prod_{i=r+1}^{2r} (x + m_i), \quad m_1 + \cdots + m_r = m_{r+1} + \cdots + m_{2r}, \] (2)
\[ D = \{(m_1, \ldots, m_{2r}) \in \mathbb{Z}^{2r} : 0 < m_i \leq h, m_i \neq m_{2r} \text{ if } i \neq 2r\}. \]

By [1], it suffices to prove

Theorem 2. If \( r = 4 \), \( q \) is a prime power, \( \chi \) is primitive and \( h \leq q^{\frac{1}{2r}} \), then
\[ \sum_{\substack{m \in D}} \left| \sum_{0 \leq x < q} \chi(f_1(x)) \overline{\chi(f_2(x))} \right| \ll q^{1+\varepsilon} h^r. \] (3)
2 On Complete Character Sums

In this section we study the complete sums
\[ \sum_{0 \leq x < q} \chi(P(x))\bar{\chi}(Q(x)), \]
where \( \chi \) is primitive, \( P(x) \) and \( Q(x) \in \mathbb{Z}[x] \).

Let \( | \cdot |_p \) be the normalized exponential valuation of the \( p \)-adic number field \( \mathbb{Q}_p \).

Let
\[ R = P/Q, \quad q | \alpha_p, \]
A = \{0 \leq x < q : (q, P(x)Q(x)) = 1\},

\[ A(p, i, \theta) = \left\{ x \in A : \left| \frac{R^{(j)}(x)}{j!} \right|_p \geq \frac{\alpha_p}{j+1} (j < i), \left| \frac{R^{(i)}(x)}{i!} \right|_p = \theta \right\}. \]

Our result concerning (4) is the following theorem.

**Theorem 3.** If \( \chi \) is primitive and
\[ \theta_p < \frac{\alpha_p}{i_p + 1} \quad \text{when} \quad i_p < 4, \]
then
\[ \sum_{\chi \in \Lambda(p,x,q)} \chi(R(x)) \ll \left( \prod_{i_p=1} p^{\alpha_p} \right)^{1/2} \left( \prod_{i_p=2} p^{\alpha_p+\theta} \right)^{1/2} \left( \prod_{i_p=3} \omega_p(\theta_p) \right) \left( \prod_{i_p=4} \min(p^{\alpha_p}, p^{[2\alpha_p]+\theta}) \right), \]

where
\[ \omega_p(\theta) = \begin{cases} p^{[\frac{1}{3}\alpha_p]} & \text{if } \theta = 0, \\ p^{p^{\alpha_p-\left[\frac{3\alpha_p}{2}\right]+\theta}} + p^{\frac{1}{3}\alpha_p+\left[\frac{3\alpha_p}{2}\right]+\theta} & \text{if } 2 \left[ \frac{\alpha_p - \theta + 2}{3} \right] < \left[ \frac{2}{3}\alpha_p \right], \\ p^{p^{\alpha_p-\left[\frac{3\alpha_p}{2}\right]+\theta}} & \text{otherwise.} \end{cases} \]

The proof of Theorem 3 needs the following lemmas.

**Lemma 1.** Let \( \chi \) be of conductor \( p^\alpha \) with \( p \) prime. Suppose that \( p \nmid a_0, \frac{\alpha}{2} \leq \theta \leq \alpha - 1 \). Then
\begin{align*}
\text{(I)} & \quad \sum_{0 \leq y < p^{\alpha-\theta}} \chi(a_0 + a_1 p^\theta y) = 0 \quad \text{if } p^{\alpha-\theta} \nmid a_1 \\
\text{(II)} & \quad \sum_{0 \leq y < p^{\alpha-\theta}} \chi(a_0 + a_1 p^\theta y + a_2 p^\theta y^2) \ll p^{\frac{\alpha \theta}{2}} \quad \text{if } p \nmid (a_1, a_2). 
\end{align*}

**Proof.** It is easy to see that \( \chi(1 + ap^\theta y) \) is an additive character of conductor \( p^{\alpha-\theta} \) with respect to \( y \) when \( p \nmid a \). So (I) follows from the orthogonality and (II) follows from the classical estimate for Gaussian sums.

**Lemma 2.** Let \( R(x) = P(x)/Q(x) \in Q(x) \). Suppose that \( p \nmid P(x_0)Q(x_0) \). Then
\[ R(x_0 + y) = \sum_{i=0}^\infty \frac{R^{(i)}(x_0)}{i!} y^i \in \mathbb{Z}_p[[y]]. \]