The Lax Representation and Darboux Transformation for Constrained Flows of the AKNS Hierarchy

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Abstract. By using a general scheme for decomposing a zero-curvature equation into two commuting $x$- and $t_n$- finite-dimensional integrable Hamiltonian systems (FDIHS), a systematic deduction of the Lax representation for all constrained flows of the AKNS hierarchy from the adjoint representation of the two auxiliary linear problems is presented. The Darboux transformation for these FDIHSs is derived.

Keywords. Constrained flow, Lax representation, Adjoint representation, Darboux transformation

1 Introduction

In a series of papers[^1-5^], a method of decomposing each equation in a hierarchy of zero-curvature equations into two commuting $x$- and $t_n$- finite-dimensional integrable Hamiltonian systems (FDIHS) was developed. It is an important problem to find the Lax representation for all these FDIHSs. Using a general scheme for higher-order decompositions of zero-curvature equations[^4,5^], we presented a unified approach to finding the Lax representation for $x$- and $t_n$- FDIHSs directly from the adjoint representation of the two auxiliary linear problems, respectively, in [6]. In this paper the approach is developed to construct the Lax representation for the FDIHSs related to the AKNS hierarchy. It is well known that the Darboux transformation provides a powerful tool to obtain solutions for equations. After having found the Lax representation, we construct the Darboux transformation for these FDIHSs.

2 The Lax Representation for Constrained Flows

Let us briefly describe the zero-curvature representation for the AKNS hierarchy associated with the following eigenvalue problem[^7^],

$$
\phi_x = U(u, \lambda)\phi = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}.
$$

(2.1)

First, we solve the stationary zero-curvature equation[^8,9^]

$$
V_x = [U, V] \equiv UV - Vu,
$$

(2.2)
with
\[ V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{m=0}^{\infty} V_m \lambda^{-m} = \sum_{m=0}^{\infty} \begin{pmatrix} a_{m}(u) & b_{m}(u) \\ c_{m}(u) & -a_{m}(u) \end{pmatrix} \lambda^{-m}. \]

Eq (2.2) is called the adjoint representation of (2.1), and leads to
\[ a_0 = -1, \quad b_0 = c_0 = a_1 = 0, \quad b_1 = q, \quad c_1 = r, \]
\[ \begin{pmatrix} c_{m+1} \\ b_{m+1} \end{pmatrix} = L \begin{pmatrix} c_m \\ b_m \end{pmatrix} = L^m \begin{pmatrix} r \\ q \end{pmatrix}, \quad a_{m,x} = q c_m - r b_m, \quad (2.3) \]
where
\[ L = \frac{1}{2} \begin{pmatrix} D - 2r D^{-1} q & 2r D^{-1} r \\ -2q D^{-1} q & -D + 2q D^{-1} r \end{pmatrix}, \quad D = \frac{\partial}{\partial x}, \quad D^{-1} D = DD^{-1} = 1. \quad (2.4) \]

Secondly, take
\[ \phi_{t_n} = V^{(n)}(u, \lambda) \phi = \sum_{m=0}^{n} V_m \lambda^{-m-1} \phi, \quad (2.5) \]
and set
\[ \phi_{t_n} = V^{(n)}(u, \lambda) \phi = \sum_{m=0}^{n} V_m \lambda^{-m-1} \phi. \quad (2.6) \]

Then the compatibility condition of (2.1) and (2.6) gives rise to a zero-curvature equation
\[ \frac{1}{D - 2r D^{-1} q} + \frac{2r D^{-1} r}{-2q D^{-1} q} - \frac{D + 2q D^{-1} r}{-D + 2q D^{-1} r} = 0, \quad (2.9) \]
which yield the AKNS hierarchy\[7-9],[
\[ u_{t_n} = \phi_{t_n} + [U, V^{(n)}] = 0, \quad n = 1, 2, \ldots, \quad (2.10) \]
where \( \phi_{t_n} \) stands for the variational derivative and
\[ J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad H_m = \frac{2}{m} a_{m+1}. \quad (2.11) \]

It is known\[5,8,10\] that the \( V \) determined by (2.2) under (2.8) also satisfies the adjoint representation of (2.6):
\[ V_t = [V^{(n)}, V]. \quad (2.12) \]
For \( N \) distinct \( \lambda_j \), as we argued in [2-5], the following system
\[ \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_x = U(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} = \begin{pmatrix} -\lambda_j & q \\ r & \lambda_j \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j = 1, \ldots, N, \quad (2.10a) \]
with (for a fixed \( k \))
\[ \frac{\delta H_{k+1}}{\delta u} - \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = 0, \quad (2.10b) \]
is invariant under all flows of (2.8). So (2.10) is expected to be completely integrable, and is called the \( \lambda \)-constrained flows of (2.8).