Differential Geometry of Tensor Product Immersions II

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Abstract: In the first part of this series, we prove that the tensor product immersion $f_1 \otimes \cdots \otimes f_{2k}$ of $2k$ isometric spherical immersions of a Riemannian manifold $M$ in Euclidean space is of $k$-type with $k \geq k$ and classify tensor product immersions $f_1 \otimes \cdots \otimes f_{2k}$ which are of $k$-type. In this article we investigate the tensor product immersions $f_1 \otimes \cdots \otimes f_{2k}$ which are of $(k+1)$-type. Several classification theorems are obtained.

Key words: Type number, tensor product immersion, submanifolds of finite type, Veronese immersion


1. Introduction

Let $V$ and $W$ be two vector spaces over the field of real numbers. Then we have the notion of the tensor product $V \otimes W$. If $V$ and $W$ are inner product spaces with their inner products given respectively by $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$, then $V \otimes W$ is also an inner product space with inner product defined by

$$\langle v \otimes w, x \otimes y \rangle = (v, x)_V (w, y)_W$$

(1.1)

Let $E^m$ denote the $m$-dimensional Euclidean space with the canonical Euclidean inner product. Then, with respect to the inner product defined above, $E^m \otimes E^m'$ is isometric to $E^{mm'}$. By applying this algebraic notion, we have the notion of tensor product map $f \otimes h : M \to E^m \otimes E^m' \cong E^{mm'}$ associated with any two maps $f : M \to E^m$ and $h : M \to E^m'$ of a given Riemannian manifold $(M, g)$ defined as follows:

$$(f \otimes h)(p) = f(p) \otimes h(p) \in E^m \otimes E^m', \quad \forall p \in M.$$  

(1.2)

An immersion $f : M \to E^m$ is said to be spherical if $f(M)$ is contained in a hypersphere of $E^m$ centered at the origin of $E^m$. Denote by $S(M)$ the set of all spherical immersions from an $n$-dimensional Riemannian manifold $(M, g)$ into Euclidean spaces. Then $\otimes$ is a binary operation on $S(M)$. Hence, if $f : M \to E^m$ and $h : M \to E^{m'}$ are immersions belonging to $S(M)$, their tensor product immersion $f \otimes h : M \to E^{m+m'}$ is an immersion in $S(M)$, called the tensor product immersion of $f$ and $h$. Similarly, if $f_i : M \to E^m_i$, $i = 1, \ldots, t$, are $t$ immersions belonging to $S(M)$, one has the tensor product immersion $f_1 \otimes \cdots \otimes f_t$ of $f_1, \ldots, f_t$.

A smooth map $x : M \to E^m$ from a Riemannian manifold $M$ into a Euclidean $m$-space $E^m$ is said to be of finite type (resp., $k$-type) if it can be decomposed into...
a sum of finitely many (resp., \( k \), not counting a constant function) eigenfunctions of the Laplacian \( \Delta \) of \( M \) from different eigenspaces, i.e., we have

\[
x = x_0 + x_1 + \ldots + x_k \quad (1.3)
\]

where \( x_0 \) is a constant vector and \( \Delta x_i = \lambda_i x_i, \ i = 1, 2, \ldots, k \), where \( \lambda_1, \ldots, \lambda_k \) are \( k \) different eigenvalues of Laplacian of \( M \) and \( \Delta \) acts on a vector function componentwise. In that case we have \( P(\Delta)(x - x_0) = 0 \), where

\[
P(T) = \prod_{i=1}^{k}(T - \lambda_i). \quad (1.4)
\]

If \( x \) cannot be represented as a finite sum (1.3), it is said to be of infinite type. The type number is a natural and important invariant associated with arbitrary maps and immersions of Riemannian manifolds into Euclidean spaces.

In the first part of this series ([2]), we proved the following

**Theorem A.** Let \( f_i : M \to S^{m_i-1}(r_i) \subseteq E^{m_i}, \ i = 1, \ldots, 2k \), be \( 2k \) spherical isometric immersions of an \( n \)-dimensional Riemannian manifold \( M \) into Euclidean spaces. Then (i) the tensor product immersion \( f_1 \otimes \cdots \otimes f_{2k} \) is of \( \ell \)-type with \( \ell \geq k \) and (ii) \( f_1 \otimes \cdots \otimes f_{2k} \) is of \( k \)-type if and only if \( M \) is isometric to an open portion of an ordinary \( n \)-sphere and the isometric immersions \( f_i' : M \to S^{m_i-1}(r_i) \), induced from \( f_i \), are totally geodesic.

In this paper, first we prove the following

**Theorem 1.** Let \( f : S^n(r) \to E^{n+1} \) be an ordinary imbedding of an \( n \)-sphere of radius \( r \) into \( E^{n+1} \) and \( h : S^n(r) \to S^{n(n+3)/2-1}(a) \subseteq E^{n(n+3)/2} \) be a Veronese immersion. Then, for any open subset \( M \) of \( S^n(r) \), the tensor product immersion \( f_1 \otimes \cdots \otimes f_{2k} \), restricted to \( M \), is mass-symmetric and of \( (k + 1) \)-type, where \( f_1 = \ldots = f_{2k-1} = f \) and \( f_{2k} = h \).

Next, we investigate tensor product immersions \( f_1 \otimes \cdots \otimes f_{2k} \) which are mass-symmetric and of \( (k + 1) \)-type. More precisely, we obtain the following results.

**Theorem 2.** Let \( f_i : M \to S^{m_i-1}(r_i) \subseteq E^{m_i}, \ i = 1, \ldots, 2k \ (k > 1) \) be \( 2k \) spherical isometric immersions of an \( n \)-dimensional Riemannian manifold \( M \) into Euclidean spaces. If the tensor product immersion \( f_1 \otimes \cdots \otimes f_{2k} \) is mass-symmetric and of \( (k + 1) \)-type, then each \( f_i \) has parallel mean curvature vector and parallel second fundamental form.

**Theorem 3.** Let \( f_i : M \to S^{m_i-1}(r_i) \subseteq E^{m_i}, \ i = 1, \ldots, 2k \ (k > 1) \) be \( 2k \) spherical isometric immersions of an \( n \)-dimensional Einsteinian manifold \( M \) into Euclidean spaces. If the tensor product immersion \( f_1 \otimes \cdots \otimes f_{2k} \) is mass-symmetric and of \( (k + 1) \)-type, then each \( f_i \) is of \( 1 \)-type.

**Theorem 4.** Let \( f_i : M \to S^{m_i-1}(r_i) \subseteq E^{m_i}, \ i = 1, \ldots, 2k \ (k > 1) \) be \( 2k \) spherical isometric immersions of an \( n \)-dimensional Riemannian manifold \( M \) into Euclidean spaces. If the tensor product immersion \( f_1 \otimes \cdots \otimes f_{2k} \) is mass-symmetric and of \( (k + 1) \)-type and if \( f_1 = f_2 \), then (1) \( M \) is an open portion of an \( n \)-sphere, (2) exactly one of \( f_1, \ldots, f_{2k} \) is given by a Veronese immersion and the remaining immersions are given by the ordinary immersions of an ordinary \( n \)-sphere.