Linear Structures on the Collections of Minimal Surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$

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Abstract: The collection of 'minimal herissons' in $\mathbb{R}^3$ is endowed with a vector space structure. The existence of this structure is related to the fact that null curves in $\mathbb{C}^3$ are described by a single map from the étale space of the sheaf of germs of holomorphic sections of the line bundle of degree 2 over $\mathbb{P}^1$ to $\mathbb{C}$, which is linear on stalks. There is an analogous construction for null curves in $\mathbb{C}^4$. This gives a similar class of minimal surfaces in $\mathbb{R}^4$.

Key words: Minimal surface, null holomorphic curve
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Introduction

In [6], a vector space structure is described on a certain class of minimal surfaces in $\mathbb{R}^3$ which the authors refer to as 'minimal herissons'. Our object in this note is to clarify the description of this structure and explain why a similar structure exists in dimension 4. We show that the existence of this structure is closely related to the fact that null curves in $\mathbb{C}^3$ are described by a single map from the étale space of the sheaf of germs of holomorphic sections of the line bundle of degree 2 over $\mathbb{P}^1$ to $\mathbb{C}$, which is linear on stalks. We outline an analogous construction for null curves in $\mathbb{C}^4$.

We use standard constructions and facts from minimal surface theory and algebraic geometry, see [1], [2], [4] and [5] for further details.

Linear Structures

Recall that a branched minimal surface in $\mathbb{R}^n$ is described by the real part of a null holomorphic curve in $\mathbb{C}^n$. If $\omega, \nu : M \rightarrow \mathbb{C}^n$ are null then $\langle \omega' + \nu', \omega' + \nu' \rangle = 2\langle \omega', \nu' \rangle$ is the obstruction to the nullity of $\omega + \nu$, and hence the conformality of $\text{Re}(\omega) + \text{Re}(\nu)$.

For $n = 3$, the Gauss map $\gamma_{\omega}$, of a null curve $\omega : M \rightarrow \mathbb{C}^3$, takes values on the quadric curve $Q_1 \simeq \mathbb{P}^1$, and (local) reparameterisation by Gauss maps removes the obstruction because it results in the linear dependence of the tangent (null) vectors being summed. More precisely, for $U \subset \mathbb{C}$, open, consider the pairs $(\omega, V)$, where $\omega : V \rightarrow \mathbb{C}^3$ is null and such that $\gamma_{\omega}(V) = U$ and $\gamma_{\omega}^{-1}$ exists on $U$. Let $\bar{\gamma}(\zeta) = \omega \circ \gamma_{\omega}^{-1}$ and observe that $\gamma_{\omega}(\zeta) = \zeta$. For pairs $(\omega, V)$, $(\nu, V')$ addition can be defined, over $U$, using: $\zeta \mapsto \bar{\gamma}(\zeta) + \bar{\nu}(\zeta)$, so that the resulting sum describes a null curve and the real part gives a minimal surface in $\mathbb{R}^3$. This underlies the vector space structure used in [6].
Let $\mathcal{O}(2) \rightarrow \mathbb{P}_1$ denote the holomorphic line bundle of degree 2 and let $\pi : \text{Sp}e[\mathcal{O}(2)] \rightarrow \mathbb{P}_1$ be the étale space of the sheaf of germs of local holomorphic sections. There is a canonical map $\Omega : \text{Sp}e[\mathcal{O}(2)] \rightarrow H^0(\mathbb{P}_1, \mathcal{O}(2)) \cong \mathbb{C}^3$ which is given on stalks by sending a germ to its 2-jet. (A conformal structure on $H^0(\mathbb{P}_1, \mathcal{O}(2))$ is determined by the cone of global sections that possess a double root on $\mathbb{P}_1$ and there is a canonical identification between $\mathbb{P}_1$ and the quadric curve of null lines in $H^0(\mathbb{P}_1, \mathcal{O}(2))$ given by $\zeta \rightarrow \{\text{sections with a double root at } \zeta\}$.) In [7], we proved:

**Theorem.** $\Omega : \text{Sp}e[\mathcal{O}(2)] \rightarrow H^0(\mathbb{P}_1, \mathcal{O}(2))$ is a null holomorphic curve which describes all non-linear null curves in $\mathbb{C}^3$. Its Gauss map is given by projection: $\gamma_1 = \pi$.

Note that $\text{Re}(\Omega) : \text{Sp}e[\mathcal{O}(2)] \rightarrow \mathbb{R}^3$ describes all non-planar minimal surfaces in $\mathbb{R}^3$.

Now recall that the stalks of $\pi : \text{Sp}e[\mathcal{O}(2)] \rightarrow \mathbb{P}_1$ are (infinite dimensional) vector spaces over $\mathbb{C}$. The vector space structure described above is simply a manifestation of the following:

**Fact.** $\Omega : \text{Sp}e[\mathcal{O}(2)] \rightarrow H^0(\mathbb{P}_1, \mathcal{O}(2))$ is linear on stalks.

This follows immediately from the fact that the 2-jet of a sum of germs at a point is simply the sum of the 2-jets of the germs.

**Remark.** This gives the most transparent picture of the linear structure on the collection of null curves in $\mathbb{C}^3$.

A similar picture emerges in dimension 4 in the following way. Recall that the quadric surface $Q_2$ is doubly ruled, giving $Q_2 \cong \mathbb{P}_1 \times \mathbb{P}_1$. The totally isotropic 2-dimensional subspaces of $\mathbb{C}^4$ form 2 families, the ‘$\alpha$-planes’ and the ‘$\beta$-planes’, which are parameterised by the factors. A null line in $\mathbb{C}^4$ lies on the intersection of an $\alpha$-plane and $\beta$-plane, which are thus uniquely determined. See [1] for further details. It follows that the Gauss map of a null curve $\omega : M \rightarrow \mathbb{C}^4$ splits: $\gamma_\omega = (\gamma_{\omega 1}, \gamma_{\omega 2})$, where $\gamma_{\omega 1}$ describes the $\alpha$-planes determined by $\gamma_\omega$ and $\gamma_{\omega 2}$ the $\beta$-planes.

Let $\mathcal{O}(1) \rightarrow \mathbb{P}_1$ be the holomorphic line bundle of degree 1 and $\pi : \text{Sp}e[\mathcal{O}(1) \oplus \mathcal{O}(1)] \rightarrow \mathbb{P}_1$ be the étale space of the sheaf of germs of local holomorphic sections of the rank 2 bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{P}_1$. There exists a canonical map $\omega : \text{Sp}e[\mathcal{O}(1) \oplus \mathcal{O}(1)] \rightarrow H^0(\mathbb{P}_1, \mathcal{O} \oplus \mathcal{O}(1)) \cong \mathbb{C}^4$ which is given on stalks by sending a germ to its 1-jet. (A conformal structure on $H^0(\mathbb{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ is given by the cone of global sections that vanish somewhere on $\mathbb{P}_1$. $\mathbb{P}_1$ is identified with the set of $\alpha$-planes in $H^0(\mathbb{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ by: $\zeta \rightarrow \{\text{sections that vanish at } \zeta\}$.) In [8], we prove the following:

**Theorem.** $\omega : \text{Sp}e[\mathcal{O}(1) \oplus \mathcal{O}(1)] \rightarrow H^0(\mathbb{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ is a null holomorphic curve that describes all null curves in $\mathbb{C}^4$ that do not lie on an $\alpha$-plane. The first factor of its Gauss map is given by projection: $\gamma_{\omega 1} = \pi$.

Note that $\text{Re}(\omega) : \text{Sp}e[\mathcal{O}(1) \oplus \mathcal{O}(1)] \rightarrow \mathbb{R}^4$ describes (essentially) all minimal surfaces in $\mathbb{R}^4$.

The analogue of the linear structure in dimension 4 is the following:

**Fact.** $\omega : \text{Sp}e[\mathcal{O}(1) \oplus \mathcal{O}(1)] \rightarrow H^0(\mathbb{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ is linear on stalks.

One can describe the analogue for null curves in $\mathbb{C}^4$ of the local construction in