Twistless KAM Tori

Giovanni Gallavotti

CNRS-CPT Luminy, case 907, F-13288 Marseille, France

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Abstract: A self-contained proof of the KAM theorem in the Thirring model is discussed.

I shall particularize the Eliasson method, [E], for KAM tori to a special model, of great interest, whose relevance for the KAM problem was pointed out by Thirring, [T] (see [G] for a short discussion of the model). The idea of exposing Eliasson’s method through simple particular cases appears in [V], where results of the type of the ones discussed here, and more general ones, are announced.

The connection between the methods of [E] and the tree expansions in the renormalization group approaches to quantum field theory and many body theory can be found also in [G]. The connection between the tree expansions and the breakdown of invariant tori is discussed in [PV].

The Thirring model is a system of rotators interacting via a potential. It is described by the hamiltonian (see [G] for a motivation of the name):

\[
\frac{1}{2} J^{-1} \dddot{A} \cdot \dddot{A} + \epsilon f(\alpha),
\]

where $J$ is the (diagonal) matrix of the inertia moments, $\dddot{A} = (A_1, \ldots, A_l) \in R^l$ are their angular momenta and $\dddot{\alpha} = (\alpha_1, \ldots, \alpha_l) \in T^l$ are the angles describing their positions: the matrix $J$ will be supposed nonsingular; but we only suppose that $\min_{j=1,\ldots,l} J_j = J_0 > 0$, and no assumption is made on the size of the twist rate $T = \min_{j=1,\ldots,l} J_j^{-1}$; the results will be uniform in $T$ (hence the name “twistless”: this is not a contradiction with the necessity of a twist rate in the general problems, see problems 1, 16, 17 in Sect. 5.11 of [G2], and [G]). We suppose $f$ to be an even
trigonometric polynomial of degree $N$:
\[ f(\tilde{\alpha}) = \sum_{0 < |\tilde{\alpha}| \leq N} f_{\tilde{\alpha}} \cos \tilde{\nu} \cdot \tilde{\alpha}, \quad f_{\tilde{\alpha}} = f_{-\tilde{\alpha}} . \] (2)

We shall consider a "rotation vector" $\tilde{\omega}_0 = (\omega_1, \ldots, \omega_l) \in \mathbb{R}^l$ verifying a strong diophantine property (see, however, the final comments) with diophantine constants $C_0, \gamma, c$; this means that:

1) \[ C_0 |\tilde{\omega}_0 \cdot \tilde{\nu}| \geq |\tilde{\nu}|^{-\tau}, \quad \tilde{\nu} \neq 0 \in \mathbb{Z}^l ; \]
2) \[ \min_{0 \leq p \leq n} |C_0 |\tilde{\omega}_0 \cdot \tilde{\nu}| - \gamma^p| > \gamma^{n+1} \text{ if } n \leq 0, 0 < |\tilde{\nu}| \leq (\gamma^{n+1})^{-\tau^{-1}} , \] (3)

and it is easy to see that the strongly diophantine vectors have full measure in $\mathbb{R}^l$ if $\gamma > 1$ and $c$ are fixed and if $\tau$ is fixed $\tau > l - 1$: we take $\gamma = 2, c = 3$ for simplicity; note that 2) is empty if $n > -3$ or $p < n + 3$. We shall set $A_0 = J\tilde{\omega}_0$. A special example can be the model $f_0(\tilde{\alpha}) = J_0(\cos \alpha_1 + \cos(\alpha_1 + \alpha_2))$.

We look for an $\varepsilon$-analytic family of motions starting at $\tilde{\alpha} = 0$ and having the form:
\[ \tilde{A} = A_0 + J(\tilde{\psi}; \varepsilon) , \quad \tilde{\alpha} = \tilde{\psi}_0 + \tilde{h}(\tilde{\psi}_0; \varepsilon) \] (4)

with $\tilde{H}(\tilde{\psi}; \varepsilon), \tilde{h}(\tilde{\psi}; \varepsilon)$ analytic in $\tilde{\psi} \in T^l$ and in $\varepsilon$ close to 0. We shall prove that such functions exist and are analytic for $|\text{Im} \psi_j| < \xi$ for $|\varepsilon| < \varepsilon_0$ with:
\[ \varepsilon_0^{-1} = b J_0^{-1} C_0^2 J_0 N^{2+l} e^{CN} e^{CN} , \] (5)

where $b, c$ are $l$-dependent positive constants, $f_0 = \max_p |f_p|$. This means that the set $\tilde{A} = A_0 + \tilde{H}(\tilde{\psi}; \varepsilon), \tilde{\alpha} = \tilde{\psi} + \tilde{h}(\tilde{\psi}; \varepsilon)$ described as $\tilde{\psi}$ varies in $T^l$ is, for $\varepsilon$ small enough, an invariant torus for Eq. (1), which is run quasi periodically with angular velocity vector $\tilde{\omega}_0$. It is a family of invariant tori coinciding, for $\varepsilon = 0$, with the torus $\tilde{A} = A_0, \tilde{\alpha} = \tilde{\psi} \in T^l$. One recognizes a version of the KAM theorem. The proof that follows simplifies the one reported in [G].

Supposing $J_0 \equiv J_1 < J_2$ the uniformity in $J_2$ (i.e. what we call the twistless property) implies that the same $\varepsilon_0$ can be used as an estimate of the radius of convergence in $\varepsilon$ of the power series describing the KAM tori with rotation vector $\tilde{\omega}_0 = (\omega_1, \omega_2)$ in the system $(2J_1^{-1})A_1^2 + \omega_2 A_2 + \varepsilon f_0(\alpha_1, \alpha_2)$, which is one of the most studied hamiltonian systems. The estimate can be improved. Note that a careful analysis of the proof of the KAM theorem also shows the uniformity in the twist rate in the case Eq. (1).

Calling $\tilde{H}^{(k)}(\tilde{\psi}), \tilde{h}^{(k)}(\tilde{\psi})$ the $k$th order coefficients of the Taylor expansion of $\tilde{H}, \tilde{h}$ in powers of $\varepsilon$ and writing the equation of motion as $\dot{\tilde{\alpha}} = J^{-1}\tilde{\alpha}$ and $\dot{\tilde{A}} = -\varepsilon \partial_{\tilde{\alpha}} f(\tilde{\alpha})$ we get immediately recursion relations for $\tilde{h}^{(k)}, h^{(k)}$. Namely $\tilde{\omega}_0 \cdot \partial^{(k)} = J^{-1} H^{(k)}$ and, for $k > 1$:
\[ \tilde{\omega}_0 \cdot \partial^{(k)} = - \sum_{m_1, \ldots, m_l) > 0} \prod_{s=1}^{l} \frac{1}{m_s} \partial_{\alpha_j} f(\tilde{\alpha})^m_{\alpha_1 \ldots \alpha_l} f(\tilde{\omega}_0 t) \prod_{s=1}^{l} \prod_{j=1}^{m_s} h^{(k)}_{s}(\tilde{\alpha}_s t) , \] (6)

where the $\sum^*$ denotes summation over the integers $k_s^* \geq 1$ with: $\sum_{s=1}^{l} \sum_{j=1}^{m_s} k_s^* = k - 1$. 