Twistless KAM Tori

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Abstract: A self-contained proof of the KAM theorem in the Thirring model is discussed.

I shall particularize the Eliasson method, [E], for KAM tori to a special model, of great interest, whose relevance for the KAM problem was pointed out by Thirring, [T] (see [G] for a short discussion of the model). The idea of exposing Eliasson’s method through simple particular cases appears in [V], where results of the type of the ones discussed here, and more general ones, are announced.

The connection between the methods of [E] and the tree expansions in the renormalization group approaches to quantum field theory and many body theory can be found also in [G]. The connection between the tree expansions and the breakdown of invariant tori is discussed in [PV].

The Thirring model is a system of rotators interacting via a potential. It is described by the Hamiltonian (see [G] for a motivation of the name):

\[ \frac{1}{2} J^{-1} \dot{A} \cdot \dot{A} + \varepsilon f(\bar{\alpha}), \]  

where \( J \) is the (diagonal) matrix of the inertia moments, \( \dot{A} = (A_1, \ldots, A_l) \in R^l \) are their angular momenta and \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_l) \in T^l \) are the angles describing their positions: the matrix \( J \) will be supposed nonsingular; but we only suppose that \( \min_{j=1,\ldots,l} J_j = J_0 > 0 \), and no assumption is made on the size of the twist rate \( T = \min J_j^{-1} \): the results will be uniform in \( T \) (hence the name “twistless”: this is not a contradiction with the necessity of a twist rate in the general problems, see problems 1, 16, 17 in Sect. 5.11 of [G2], and [G]). We suppose \( f \) to be an even

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trigonometric polynomial of degree $N$:

$$f(\vec{\alpha}) = \sum_{0 < |\vec{\alpha}| \leq N} f_{\vec{\alpha}} \cos \theta \cdot \vec{\alpha}, \quad f_{\vec{\alpha}} = f_{-\vec{\alpha}}.$$  \hspace{1cm} (2)

We shall consider a “rotation vector” $\vec{\omega}_0 = (\omega_1, \ldots, \omega_l) \in \mathbb{R}^l$ verifying a strong diophantine property (see, however, the final comments) with diophantine constants $C_0, \tau, \gamma, c$; this means that:

1) $C_0|\vec{\omega}_0 \cdot \vec{v}| \geq |\vec{v}|^{-\tau}, \quad \vec{0} \neq \vec{v} \in \mathbb{Z}^l$,

2) $\min_{0 \leq p \leq n} |C_0|\vec{\omega}_0 \cdot \vec{v}| - \gamma^p > \gamma^{n+1}$ \hspace{1cm} if \hspace{1cm} $n \leq 0$, $0 < |\vec{v}| \leq (\gamma^{n+c})^{-\tau+1}$,  \hspace{1cm} (3)

and it is easy to see that the strongly diophantine vectors have full measure in $\mathbb{R}^l$ if $\gamma > 1$ and $c$ are fixed and if $r$ is fixed $r > l - 1$; we take $\gamma = 2$, $c = 3$ for simplicity; note that 2) is empty if $n > -3$ or $p < n + 3$. We shall set $\vec{A}_0 = J\vec{\omega}_0$. A special example can be the model $f_{\vec{\alpha}}(\vec{\alpha}) = J_{\vec{\alpha}}(\cos \alpha_1 + \cos(\alpha_1 + \alpha_2))$.

We look for an $\varepsilon$-analytic family of motions starting at $\vec{\alpha} = \vec{0}$ and having the form:

$$A = \vec{A}_0 + \vec{H}(\vec{\omega}_0 t; \varepsilon), \quad \vec{\alpha} = \vec{\omega}_0 t + \vec{h}(\vec{\omega}_0 t; \varepsilon)$$  \hspace{1cm} (4)

with $\vec{H}(\vec{\psi}; \varepsilon), \vec{h}(\vec{\psi}; \varepsilon)$ analytic in $\vec{\psi} \in T^l$ and in $\varepsilon$ close to 0. We shall prove that such functions exist and are analytic for $|\text{Im} \psi_j| < \xi$ for $|\varepsilon| < \varepsilon_0$ with:

$$\varepsilon_0^{-1} = bJ_0^{-1}C_0^{-2}J_0^{-1}N^{2+l}e^{cN}e^{\xi N},$$  \hspace{1cm} (5)

where $b, c$ are $l$-dependent positive constants, $f_0 = \max_{\vec{\alpha}} |f_{\vec{\alpha}}|$. This means that the set $A = \vec{A}_0 + \vec{H}(\vec{\psi}; \varepsilon), \vec{\alpha} = \vec{\psi} + \vec{h}(\vec{\psi}; \varepsilon)$ described as $\vec{\psi}$ varies in $T^l$ is, for $\varepsilon$ small enough, an invariant torus for Eq. (1), which is run quasi periodically with angular velocity vector $\vec{\omega}_0$. It is a family of invariant tori coinciding, for $\varepsilon = 0$, with the torus $A = \vec{A}_0, \vec{\alpha} = \vec{\psi} \in T^l$. One recognizes a version of the KAM theorem. The proof that follows simplifies the one reported in [G].

Supposing $J_0 \equiv J_1 < J_2$ the uniformity in $J_2$ (i.e. what we call the twistless property) implies that the same $\varepsilon_0$ can be used as an estimate of the radius of convergence of $\varepsilon$ in the power series describing the KAM tori with rotation vector $\vec{\omega}_0 = (\omega_1, \omega_2)$ in the system $(2J_1^{-1})A_1^2 + \omega_2A_2 + \varepsilon f_0(\alpha_1, \alpha_2)$, which is one of the most studied hamiltonian systems. The estimate can be improved. Note that a careful analysis of the proof of the KAM theorem also shows the uniformity in the twist rate in the case Eq. (1).

Calling $\vec{H}(\vec{\psi})$, $\vec{h}(\vec{\psi})$ the $k$th order coefficients of the Taylor expansion of $\vec{H}, \vec{h}$ in powers of $\varepsilon$ and writing the equation of motion as $\vec{\dot{\alpha}} = J^{-1}\vec{A}$ and $\vec{\dot{\alpha}} = -\varepsilon \partial_{\vec{\alpha}} f(\vec{\alpha})$ we get immediately recursion relations for $\vec{H}(\vec{\psi}), \vec{h}(\vec{\psi})$. Namely $\vec{\omega}_0 \cdot \partial_{\vec{\alpha}} \vec{H}(\vec{\psi}) = J^{-1} \vec{H}(\vec{\psi})$ and, for $k > 1$:

$$\vec{\omega}_0 \cdot \partial \vec{H}(\vec{\psi}) = - \sum_{m_1, \ldots, m_l, |n| > 0} \frac{1}{\prod_{s=1}^{m_l} m_s!} \partial_{\alpha_{j}} \partial_{\alpha_{j}}^{m_1 \ldots m_l} f(\vec{\omega}_0 t) \cdot \sum_{s=1}^{l} \sum_{j=1}^{m_s} h_{s,j}^{(\psi)}(\vec{\omega}_0 t),$$  \hspace{1cm} (6)

where the $\sum^*$ denotes summation over the integers $k_s^2 \geq 1$ with: $\sum_{s=1}^{l} \sum_{j=1}^{m_s} k_s^2 = k - 1$. 